2.2 – The Inverse of a Matrix

Math 220
Warnock - Class Notes

Remember that the multiplicative inverse or reciprocal of a number, say 7 is 1/7 or 7^{-1}. The actual definition of this is that

$$7^{-1} \cdot 7 = 1 \qquad 7 \cdot 7^{-1} = 1$$

An $(n \times n)$ matrix A is called invertible if there is a matrix *C* such that

$$CA = I \qquad \text{and} \qquad AC = I$$

($I = I_n$ is the $n \times n$ identity matrix.)

* Here, C is called the Inverse of A. Is C unique?

Prove Inverse of A is unique.

Let B be another inverse of A.

$B = B(I) = B(AC) = (BA)C = IC = C$

$\therefore$ Inverse is Unique

Yes, so denote the inverse with A^{-1} and

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

A matrix that is **NOT** invertible is called a singular matrix while a matrix that **IS** invertible is called a nonsingular matrix.

Ex 1: If $A = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix}$, verify that $C = A^{-1}$.

$$AC = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 10-9 & 6-6 \\ -15+15 & -9+10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

$$CA = \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 10-9 & 15-15 \\ -6+6 & -9+10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

So $C = A^{-1}$

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$\boxed{A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

If $ad - bc = 0$, then A is not invertible.

This value $ad - bc$ is called the determinant and we write

$$\det A = ad - bc$$

So theorem 4 states that 2x2 matrix is invertible iff $\det A \neq 0$.

(if and only if)

Ex 2: Find the inverse of $A = \begin{bmatrix} 3 & -7 \\ 2 & 5 \end{bmatrix}$.

$\det A = 15 - (-14) = 29$

$$A^{-1} = \frac{1}{29}\begin{bmatrix} 5 & 7 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5/29 & 7/29 \\ -2/29 & 3/29 \end{bmatrix}$$

Theorem 5

✱ If A is an invertible $n \times n$ matrix, then for each **b** in $\mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

Proof:

Existence

Let $\vec{b} \in \mathbb{R}^n$, does $A\vec{x} = \vec{b}$ have a solution?

A is invertible, so A^{-1} exists, let $\vec{x} = A^{-1}\vec{b}$

$A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b} \Rightarrow$ soln exists

Uniqueness

let $\vec{u}$ be another solution to $A\vec{x} = \vec{b}$

$A\vec{u} = \vec{b} \Rightarrow A^{-1}A\vec{u} = A^{-1}\vec{b}$

$\Rightarrow I\vec{u} = A^{-1}\vec{b}$

$\vec{u} = A^{-1}\vec{b} = \vec{x}$

$\Rightarrow$ soln unique

Ex 3: Use the inverse of the matrix $A = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$ from Ex 1 $\left(A^{-1} = \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} \right)$

to solve the system $\begin{aligned} -2x_1 - 3x_2 &= 5 \\ 3x_1 + 5x_2 &= -7 \end{aligned}$

$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$

$\begin{bmatrix} -2 & -3 & 5 \\ 3 & 5 & -7 \end{bmatrix} \xrightarrow{\text{rref}}$

$\vec{x} = \begin{bmatrix} -5 & -3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -7 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

Theorem 6

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

* b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

* c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$(AB)^T = B^T A^T$

$$(A^T)^{-1} = (A^{-1})^T$$

Proofs: a) Find $CA^{-1} = I$, $A^{-1}C = I$, Check A. Yes. so $(A^{-1})^{-1} = I$

$AA^{-1} = I \quad A^{-1}A = I$

b) $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = (AI)A^{-1} = AA^{-1} = I$ ✓

Similarly $(B^{-1}A^{-1})(AB) = I \quad \therefore (AB)^{-1} = B^{-1}A^{-1}$

c) $A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

$(AB)^{-1} = B^{-1}A^{-1}$

From Theorem 6b, we can extrapolate to the following.

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order. $(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$

(Read pages 108-109 on Elementary Matrices)

We are going to look at finding the inverse of a matrix with a slightly different approach than this text.

If an $n \times n$ matrix *A* has an inverse, let's call that matrix *B.* Then

$$AB = I$$

This can be written as

$$\left[A\vec{b}_1 \;\; A\vec{b}_2 \;\; A\vec{b}_3 \cdots A\vec{b}_n\right] = \left[\vec{e}_1 \;\; \vec{e}_2 \;\; \vec{e}_3 \cdots \vec{e}_n\right]$$

We can think of this as many systems, where each solution forms the columns vectors of our matrix B.

$A\vec{b}_1 = \vec{e}_1$ $\quad A\vec{b}_n = \vec{e}_n$

$A\vec{b}_2 = \vec{e}_2$

$$\left[A \,\vdots\, \vec{e}_1 \;\; \vec{e}_2 \;\; \vec{e}_3 \cdots \vec{e}_n\right]$$

$$\left[A \,\vdots\, I\right]$$

We could solve each one of these individually, or stack them all together.

Ex 4: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -1 & 10 \end{bmatrix}$.

$\vec{b}_1$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 5 & 4 & 0 \\ 1 & -1 & 10 & 0 \end{array}\right]$

$\vec{b}_2$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 4 & 1 \\ 1 & -1 & 10 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & -1 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{rref}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 54 & -23 & -7 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right] \Rightarrow \boxed{A^{-1} = \begin{bmatrix} 54 & -23 & -7 \\ -16 & 7 & 2 \\ -7 & 3 & 1 \end{bmatrix}}$$

Theorem 7
An $n \times n$ matrix *A* is invertible if and only if *A* is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces *A* to I_n also transforms I_n into A^{-1}.

$$[A \,\vdots\, I] \xrightarrow{rref} [I \,\vdots\, A^{-1}]$$

Algorithm for Finding A^{-1}
Row reduce the augmented matrix $[A \quad I]$. If *A* is row equivalent to *I*, then $[A \quad I]$ is row equivalent to $[I \quad A^{-1}]$. Otherwise, *A* does not have an inverse.

Ex 5: Find the inverse of the matrix $A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$, if it exists.

(Do this by hand – more practice.)

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_1+R_2 \\ -5R_1+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \\ \\ -2R_2+R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{array}\right]$$

A is not invertible