1.9 – Matrix of a Linear Transformation **Math 220**

Ex 1: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ such that $T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix}$.

Find a formula for the image of an arbitrary $\mathbf{x} \in \mathbb{R}^2$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= x_1 \vec{e}_1 + x_2 \vec{e}_2$$

$$T(\vec{x}) = T(x_1 \vec{e}_1 + x_2 \vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix} = \begin{bmatrix} 3x_1 + 0x_2 \\ 2x_1 - x_2 \\ -5x_1 + 9x_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & -1 \\ -5 & 9 \end{bmatrix} [$$

This shows us that knowing $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ can give us $T(\mathbf{x})$ for any $\mathbf{x}$.

$$T(\mathbf{x}) = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}$$

Theorem 10

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose j th column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j th column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)] \tag{3}$$

This Matrix A is called the standard matrix for the linear transformation T.

Ex 2: Find the standard matrix *A* for the contraction transformation $T(\mathbf{x}) = \frac{1}{2}\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^2$.

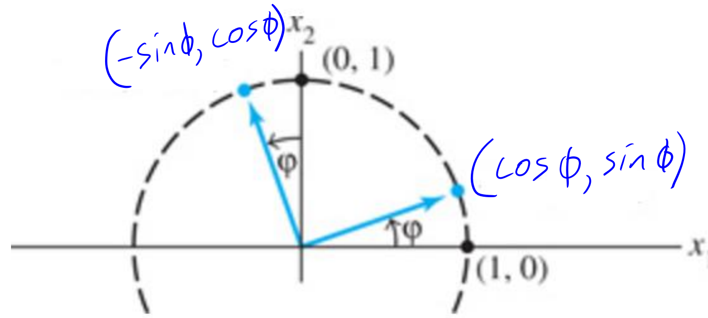
$T(\vec{e}_1) = T\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1/2\\0\end{bmatrix}$, $T(\vec{e}_2) = T\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1/2\end{bmatrix}$

$$A = \begin{bmatrix}1/2 & 0\\0 & 1/2\end{bmatrix}$$

Ex 3: Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through the angle φ, with counterclockwise rotation for a positive angle (see the figure). Find the standard matrix *A* of this transformation.

$T(\vec{e}_1) = T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\phi\\\sin\phi\end{bmatrix}$

$T(\vec{e}_2) = T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\phi\\\cos\phi\end{bmatrix}$



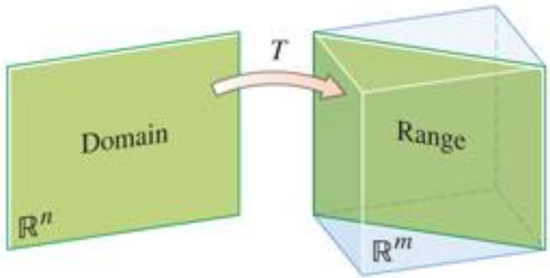
$$A = \begin{bmatrix}\cos\phi & -\sin\phi\\\sin\phi & \cos\phi\end{bmatrix}$$

Ex 4: Observe and discuss in the interactive ebook: *(also, pages 74-76)*

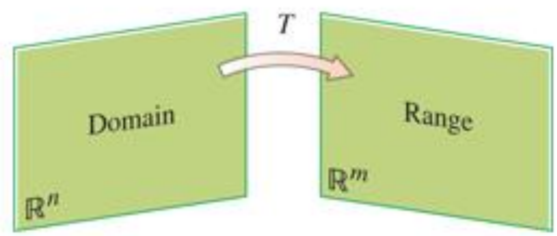
- Reflection
- Contraction & Expansion
- Shear
- Projection

Definition
A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$.

Another way of saying this is that the range of *T* is all of the codomain $\mathbb{R}^m$

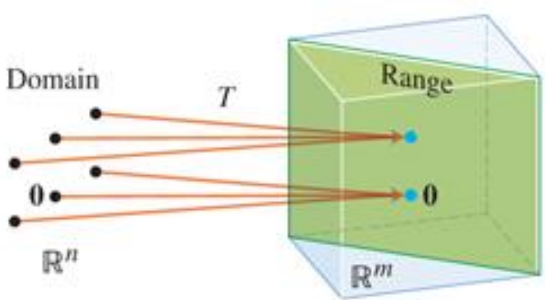


T is *not* onto $\mathbb{R}^m$

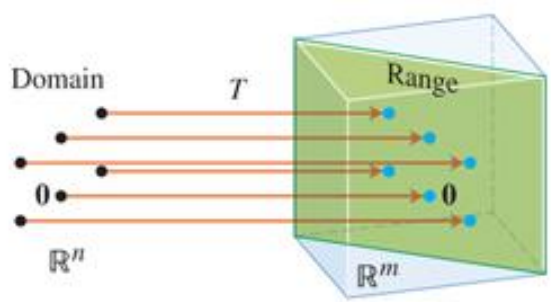


T is onto $\mathbb{R}^m$

Definition
A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$.



T is *not* one-to-one



T is one-to-one

Theorem 11
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then *T* is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Proof: $(\Rightarrow)$ Assume T is 1-to-1. $T(\vec{0}) = \vec{0}$ (LT)

$T(\vec{x}) = \vec{0} \Rightarrow \vec{x} = \vec{0}$ (1-to-1) Q.E.D.

$(\Leftarrow)$ $T(\vec{x}) = \vec{0}$ has only the trivial solution.

Assume T is not 1-to-1. Then image $\vec{b}$ comes from two vectors $\vec{u}$ & $\vec{v}$, $\vec{u} \neq \vec{v}$.

$T(\vec{u}-\vec{v}) = T(\vec{u}) - T(\vec{v}) = \vec{b} - \vec{b} = \vec{0} \Rightarrow \vec{u} - \vec{v} = \vec{0}$

$\vec{u} = \vec{v}$

Therefore, by contradiction, T is 1-to-1.

Theorem 12

Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear transformation, and let *A* be the standard matrix for *T*. Then:

a. *T* maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of *A* span $\mathbb{R}^m$;

b. *T* is one-to-one if and only if the columns of *A* are linearly independent.

Proof: a) Columns of A span $\mathbb{R}^m$

$\Leftrightarrow$ every $\vec{b} \in \mathbb{R}^n$ has a solution for $A\vec{x} = \vec{b}$

$\Leftrightarrow$ every $\vec{b}$ in $T(\vec{x}) = \vec{b}$ has a solution

$\Leftrightarrow$ T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$

b) $T(\vec{x}) = 0$ and $A\vec{x} = \vec{0}$ represent the same thing

T is 1-to1 $\Leftrightarrow$ $A\vec{x} = \vec{0}$ has only trivial solution, $\vec{0}$

$\Leftrightarrow$ Columns of A are linearly independent

T 1-to-1: A has no free variables

T is onto $\mathbb{R}^m \Leftrightarrow$ A has m pivot positions

Ex 5: Let T be the linear transformation whose standard matrix is below (2 cases). Determine whether they are "onto $\mathbb{R}^3$" and/or a one-to-one mapping.

a) $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & 5 \end{bmatrix}$

$B\vec{x} = \vec{b}$

	Why?	Why?
onto $\mathbb{R}^3$?	Yes, 3 pivot positions span $\mathbb{R}^3$	No, can't span $\mathbb{R}^3$ (only 2 vectors)
one-to-one?	No, free variables (linear dependance)	Yes, linearly independent columns (not scalar multiple) (2 vectors)

Neithers $\begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & -6 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$