<u>1.9 – Matrix of a Linear Transformation</u> Math 220

Warnock - Class Notes

Ex 1: The columns of
$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
 are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose *T* is a linear
transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ such that $T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix}$.
Find a formula for the image of an arbitrary $\mathbf{x} \in \mathbb{R}^2$
 $\overrightarrow{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = X_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + X_2 \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix}$.
 $\overrightarrow{Y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\$

This shows us that knowing $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ can give us $T(\mathbf{x})$ for any \mathbf{x} .

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\mathbf{x}$$

Theorem 10

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose j th column is the vector $T(\mathbf{e}_j)$, where \mathbf{e}_j is the j th column of the identity matrix in \mathbb{R}^n :

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)] \tag{3}$$

This Matrix A is called the <u>Standard matrix</u> <u>For the linear transformation</u> T.

Ex 2: Find the standard matrix A for the contraction transformation
$$T(\mathbf{x}) = \frac{1}{2}\mathbf{x}$$

for $\mathbf{x} \in \mathbb{R}^2$. $T(\vec{e}_1) = T[\vec{o}] = \vec{1}[\vec{o}] = [\frac{1}{2}], \quad T(\vec{e}_2) = T[\vec{o}] = \frac{1}{2}[\vec{o}] = \frac{1}{2}[\vec{$

Ex 3: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in \mathbb{R}^2 about the origin through the angle φ , with counterclockwise rotation for a positive angle (see the figure). Find the standard matrix *A* of this transformation.



Ex 4: Observe and discuss in the interactive ebook: (also, pages 74-76)

- Reflection
- Contraction & Expansion
- Shear
- Projection

Definition

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** \mathbb{R}^m if each **b** in \mathbb{R}^m is the image of *at least one* **x** in \mathbb{R}^n .

Another way of saying this is that the <u>Yange</u> of T is all of the <u>codomain</u> \mathbb{R}^m



Definition

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each **b** in \mathbb{R}^m is the image of *at most one* **x** in \mathbb{R}^n .



Theorem 11

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then *T* is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

Proof: (=) Assume T is 1-to-1.
$$T(\hat{o})=\hat{o}$$
 (LT)
 $T(\hat{x})=\hat{o} \Rightarrow \hat{x}=\hat{o}$ (1-to-1) Q.E.D.
(=) $T(\hat{x})=\hat{o}$ has only the trivial solution.
Assume T is not 1-to-1. Then image \hat{b} comes
from two vectors $\hat{u} \neq \hat{v}$, $\hat{u} \neq \hat{v}$.
 $T(\hat{u}-\hat{v})=T(\hat{u})-T(\hat{v})=\hat{b}-\hat{b}=\hat{o}=\hat{p}$ $\hat{u}-\hat{v}=\hat{o}$
Therefore, by contradiction, $\hat{u}=\hat{v}$

Theorem 12

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

- a. T maps \mathbb{R}^n onto \mathbb{R}^m if and only if the columns of <u>A</u> span \mathbb{R}^m ;
- b. T is one-to-one if and only if the columns of A are linearly independent.

Proof: a) Columns of A span R^M
#D every
$$\vec{b} \in \mathbb{R}^m$$
 has a solution for $A\vec{x} = \vec{b}$
#D every \vec{b} in $T(\vec{x}) = \vec{b}$ has a solution
#D T maps \mathbb{R}^n onto \mathbb{R}^m
b) $T(\vec{x}) = 0$ and $A\vec{x} = \vec{0}$ represent the same thing
 T is $1 - tol #D A\vec{x} = \vec{0}$ has only trivial solution, $\vec{0}$
#D Columns of A are linearly independent
 T $1 - to - 1: A$ has no five variables
 T is onto $\mathbb{R}^m \neq \vec{0}$ A has m pivot positions

Ex 5: Let T be the linear transformation whose standard matrix is below (2 cases). Determine whether they are "onto \mathbb{R}^3 " and/or a one-to-one mapping.

a)
$$A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

b)
$$B = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & 5 \end{bmatrix}$$

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No, Can't span \mathbb{R}^3
(on by 2 vectors)
one-to-one? No, free variables
(linear dependence) Ves, linearly independent
(linear dependence) (not scalar multiple)
(2 vectors)
(not scalar multiple)
(2 vectors)