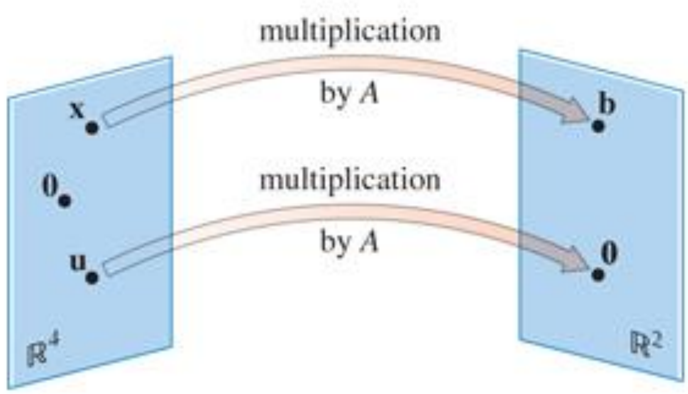
1.8 – Linear Transformations

Math 220
Warnock - Class Notes

While the matrix equation $A\vec{x} = \vec{b}$ and the vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_p\vec{a}_p = \vec{b}$ are essentially the same except for notation, there is a case where the matrix equation represents an action on a vector that isn't directly connected with a linear combination of vectors.

$$\underset{\overset{\uparrow}{A}}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\overset{\uparrow}{\mathbf{x}}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{\overset{\uparrow}{\mathbf{b}}}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}} \quad \text{and} \quad \underset{\overset{\uparrow}{A}}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\overset{\uparrow}{\mathbf{u}}}{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}} = \underset{\overset{\uparrow}{\vec{0}}}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$



Does this picture look familiar from other math you've seen? Functions for vectors

A Transformation T from $\mathbb{R}^N$ to $\mathbb{R}^M$ is a rule that assigns each vector $\mathbf{x} \in \mathbb{R}^N$ to a vector $T(\mathbf{x}) \in \mathbb{R}^M$.

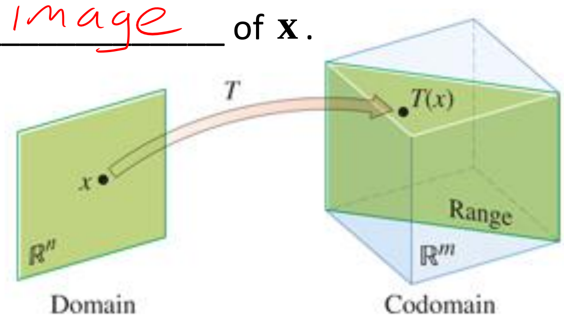
$$T : \mathbb{R}^N \to \mathbb{R}^M$$

The set $\mathbb{R}^N$ is called the Domain of *T*.

The set $\mathbb{R}^M$ is called the Codomain of *T*.

For $\mathbf{x} \in \mathbb{R}^N$, the vector $T(\mathbf{x}) \in \mathbb{R}^M$ is called the image of $\mathbf{x}$.

The set of all images $T(\mathbf{x})$ is called the Range of *T*.



Ex 1: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}, \mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix},$

define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x},$ so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a. Find $T(\mathbf{u})$, the image of **u** under the transformation *T*.

b. Find an **x** in $\mathbb{R}^2$ whose image under *T* is **b**.

c. Is there more than one **x** whose image under *T* is **b**?

d. Determine if **c** is in the range of the transformation *T*.

a) $T\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$ is the image of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ under T

b) $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ $\quad \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{array}\right] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$ $\boxed{\vec{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}}$

$T\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

c) No, no free variables

d) $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ $\quad \left[\begin{array}{cc|c} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

$\vec{c}$ is not in the range of $T(\vec{x})$

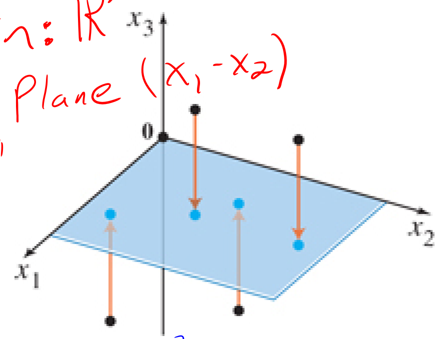
$0 \neq 1$
no solution

Ex 2: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ *projects* points in $\mathbb{R}^3$ onto the x_1, x_2-plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

Codomain: $\mathbb{R}^3$

Range: Plane $(x_1 \text{-} x_2)$ "$\mathbb{R}^2$"



$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Codomain: $\mathbb{R}^2$

Range: $\mathbb{R}^2$

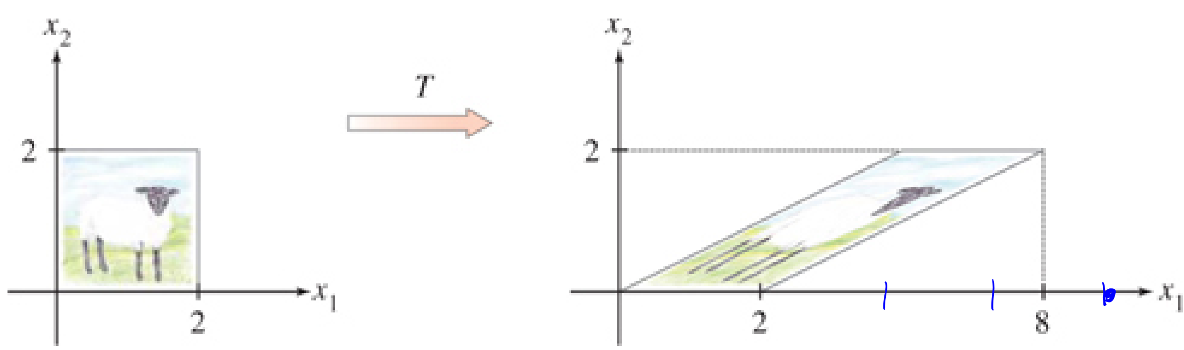
Ex 3: Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a ___shear___ ___transformation___.

For the image below, let's look at the transformations of the vectors $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$T\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$T\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$



Definition

A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all **u** in the domain of T.

Since the above properties are true for all matrices, then every <u>matrix</u> transformation is a <u>linear</u> transformation. (Though the reverse is not true.)

Furthermore, (mini proof)

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

$$T(\vec{0}) = T(0\vec{u}) = 0T(\vec{u}) = \vec{0}$$

and

$$T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

$$T(c\vec{u}+d\vec{v}) = T(c\vec{u}) + T(d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$$

for all vectors **u**, **v** in the domain of T and all scalars c, d.

The second property here actually can be generalized to

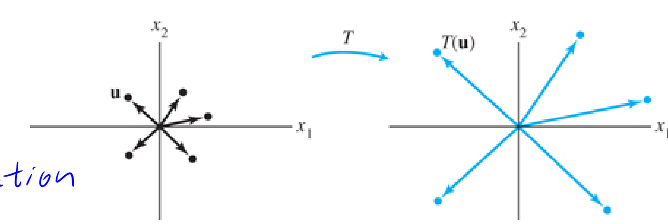
$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$$

This is referred to as a <u>superposition</u> <u>principle</u> in engineering and physics.

Ex 4: Given a scalar r, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. T is called a <u>contraction</u> when $0 \le r \le 1$ and a <u>dilation</u> when $r > 1$. Let $r = \pi$ and show that T is a linear transformation.

$$T(c\mathbf{u}+d\mathbf{v}) = \pi(c\vec{u}+d\vec{v})$$
$$= \pi c\vec{u} + \pi d\vec{v}$$
$$= c\pi\vec{u} + d\pi\vec{v}$$
$$= cT(\vec{u}) + dT(\vec{v})$$

$\Rightarrow$ Linear Transformation



Review Ex. 5 on page 68 of a Rotation Transformation.