1.7 – Linear Independence

Definition
An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

$x_1 = x_2 = \ldots = x_p = 0$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Ex 1: Determine whether the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. If not, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2x_3$

$x_2 = -x_3$

Let $x_3 = 1$

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$A\vec{x} = \vec{0}$

$$\boxed{2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}}$$

$-2\vec{v}_1 + \vec{v}_2 - \vec{v}_3 = \vec{0}$

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

$\vec{x} = \vec{0}$

Ex 2: Determine whether the columns of the matrix $A=\begin{bmatrix}1 & -2 & 1\\ 2 & 1 & -1\\ -3 & 1 & -2\end{bmatrix}$ are linearly independent.

$$\begin{bmatrix}1 & -2 & 1 & 0\\ 2 & 1 & -1 & 0\\ -3 & 1 & -2 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} \to \boxed{\text{lin ind}}$$

($\vec{x}=\vec{0}$ only solution)

Ex 3: What about?

	Linearly Dependent or Independent?	**Why?**
$\{\mathbf{v}\}$, not the zero vector	Lin Ind	$c_1\vec{v}_1=\vec{0}$; $c_1=0$ only solution
$\{\mathbf{0}\}$	Lin Dep	$c_1\vec{0}=\vec{0}$; $c_1=$ any real number
$\left\{\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}-3\\6\end{bmatrix}\right\}$	Lin Dep	$c_1\begin{bmatrix}1\\-2\end{bmatrix}+c_2\begin{bmatrix}-3\\6\end{bmatrix}=\vec{0}$; $c_1=3,\ c_2=1$; $-3\vec{v}_1=\vec{v}_2$
$\left\{\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}-3\\5\end{bmatrix}\right\}$	Lin Ind	$c_1\begin{bmatrix}1\\-2\end{bmatrix}+c_2\begin{bmatrix}-3\\5\end{bmatrix}=\vec{0}$ has only $\vec{0}$ solution; $\begin{bmatrix}1 & -3 & 0\\ -2 & 5 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$

A set of two vectors $\{\mathbf{v}_1,\mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

Theorem 7 Characterization of Linearly Dependent Sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Proof: ($\Leftarrow$) Let $\vec{v}_j$ be a linear combination of the preceding vectors

$\vec{v}_j = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_{j-1}\vec{v}_{j-1}$, $c_1, \ldots, c_{j-1}$ are not all zero

$$\vec{0} = \underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_{j-1}\vec{v}_{j-1} - \vec{v}_j + 0\vec{v}_{j+1} + 0\vec{v}_{j+2} + \ldots + 0\vec{v}_p}_{\text{nonzero solution} \Rightarrow S \text{ is linearly dependent}}$$

($\Rightarrow$) S is linearly dependent

If $\vec{v}_1 = \vec{0}$, then it is a linear combination of the others. $c_2 = c_3 = c_4 = \ldots = c_p = 0$

Suppose $\vec{v}_1 \neq \vec{0}$, there exists $c_1, c_2, \ldots, c_p$, not all zero, such that

$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \vec{0}$ Let j be the largest subscript where $c_j \neq 0$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_j\vec{v}_j + 0\vec{v}_{j+1} + \ldots + 0\vec{v}_p = \vec{0} \Rightarrow \vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \ldots - \frac{c_{j-1}}{c_j}\vec{v}_{j-1}$$

Q.E.D.

Ex 4: Given the set of vectors $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \in \mathbb{R}^3$ with $\mathbf{u}$ and $\mathbf{v}$ linearly independent, explain why vector $\mathbf{w}$ is in the plane spanned by $\mathbf{u}$ and $\mathbf{v}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

$\vec{u}$ & $\vec{v}$ are lineary indepent

Span is a plane

$\vec{w} = a\vec{u} + b\vec{v}$

$\Leftrightarrow$ $\vec{0} = -\vec{w} + a\vec{u} + b\vec{v}$

$\Leftrightarrow$ linearly dependent (nonzero solution)

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

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Proof: Let $p > n$ (more vectors than degree of the space)

$\Rightarrow$ free variables

$\Rightarrow$ $A\vec{x} = \vec{0}$ has nontrivial solutions

$\Rightarrow$ linearly dependent

Q.E.D.

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a & 0 \\ 0 & 1 & 0 & \cdots & 0 & b & 0 \\ 0 & 0 & 1 & \cdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & y & 0 \\ 0 & 0 & 0 & \cdots & 1 & z & 0 \end{bmatrix}$$

Ex 5: Using Theorem 8, create a set of vectors in $\mathbb{R}^3$ that is linearly dependent, and don't automatically make some of the vectors obvious multiples or combinations of the others.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 9 \\ 17 \end{bmatrix}, \begin{bmatrix} e \\ \pi \\ \sqrt{2} \end{bmatrix} \right\}$$

Theorem 9

If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

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Proof: $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_p\vec{v}_p = \vec{0}$ (find non-zero solution)

w.l.o.g., let $\vec{v}_1 = \vec{0}$, let $c_1 \neq 0$

(without loss of generality)

$c_1\vec{0} + 0\vec{v}_2 + 0\vec{v}_3 + \ldots + 0\vec{v}_p = \vec{0}$

$\begin{bmatrix} c_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is our non-trivial solution.

$\therefore$ linearly dependent Q.E.D.

Ex 6:

Determine by inspection if the given set is linearly dependent.

a. $\begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}$

Lin Dep, 4 vectors in $\mathbb{R}^3$ $(4>3)$

b. $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}$

Lin Dep, $\vec{0}$ is present

c. $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$

Lin Ind,
Neither is a scalar multiple of the other
(Close! Need -15 in last entry)

1.6 Exercise (we did chemistry example yesterday)

a) Find the general traffic pattern in the freeway network shown in the figure. (Flow rates are in cars/minute)

b) Describe the general traffic pattern when the road whose flow is x_4 is closed.

c) When $x_4 = 0$, what is the minimum value of x_1?

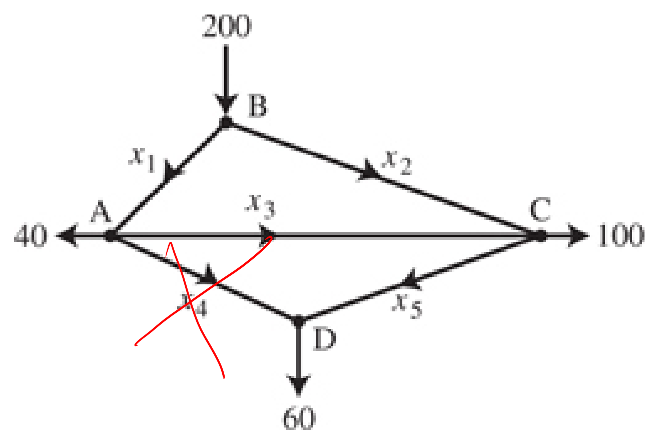
In Out

A: $x_1 = x_3 + x_4 + 40$

B: $200 = x_1 + x_2$

C: $x_2 + x_3 = 100 + x_5$

D: $x_4 + x_5 = 60$



$$\begin{aligned} x_1 \quad - x_3 - x_4 \quad &= 40 \\ x_1 + x_2 \quad &= 200 \\ x_2 + x_3 \quad - x_5 &= 100 \\ x_4 + x_5 &= 60 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix}$$

$\xrightarrow{\text{rref}}$

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) $x_1 = 100 + x_3 - x_5$

$x_2 = 100 - x_3 + x_5$

$x_4 = 60 - x_5$

b) $x_4 = 0$

$$\boxed{\begin{aligned} x_5 &= 60 \\ x_2 &= 160 - x_3 \\ x_1 &= 40 + x_3 \end{aligned}}$$

c) Min of x_1 is 40