1.5 – Solution Sets of Linear Systems

Math 220
Warnock - Class Notes

A system of linear equations is called Homogeneous if it can be written as $A\mathbf{x}=0$ Such a system always has the trivial solution $\vec{x}=\vec{0}$.

The important question is whether or not there is a non-trivial solution to a homogeneous system. $\vec{v} \neq \vec{0}$

Since there is always a trivial solution, there is only a non-trivial solution if and only if there is at least one free variable.

Ex 1: Determine whether the following has a non-trivial solution, and if so, describe the solution set.

$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 17/8 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -\frac{17}{8} x_3$

$x_2 = \frac{3}{4} x_3$

$$\vec{x} = \begin{bmatrix} -17/8\, x_3 \\ 3/4\, x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -17/8 \\ 3/4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -17 \\ 6 \\ 8 \end{bmatrix}$$

Ex 2: Describe all the solutions of the homogeneous "system".

$$3x_1 - 4x_2 + 5x_3 = 0$$

$$\begin{bmatrix} 3 & -4 & 5 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1} \begin{bmatrix} 1 & -\frac{4}{3} & \frac{5}{3} & 0 \end{bmatrix}$$

$x_1 = \frac{4}{3} x_2 - \frac{5}{3} x_3$

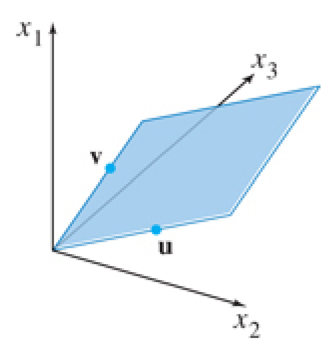
$$\vec{x} = \begin{bmatrix} \frac{4}{3}x_2 - \frac{5}{3}x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\vec{x} = x_2 \begin{bmatrix} 4/3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5/3 \\ 0 \\ 1 \end{bmatrix}$$

$x_2 = \frac{3}{4} x_1 + \frac{5}{4} x_3$

$$\vec{x} = \begin{bmatrix} x_1 \\ \frac{3}{4}x_1 + \frac{5}{4}x_3 \\ x_3 \end{bmatrix}$$

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 3/4 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 5/4 \\ 1 \end{bmatrix}$$



The previous example demonstrates how we can write solutions in Parametric Vector Form. $\mathbf{x} = s\mathbf{u} + t\mathbf{v} \quad (s, t \in \mathbb{R})$

$s = x_2, \; t = x_3$

$s = x_1, \; t = x_3$

Solutions of Nonhomogeneous Systems

Ex 3: Describe all solutions of $A\mathbf{x} = \mathbf{b}$. $A = \begin{bmatrix} 1 & 3 & 1 \\ -4 & -9 & 2 \\ 0 & -3 & -6 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}$

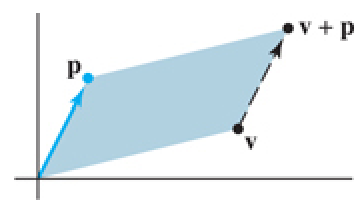
$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 5x_3 - 2$

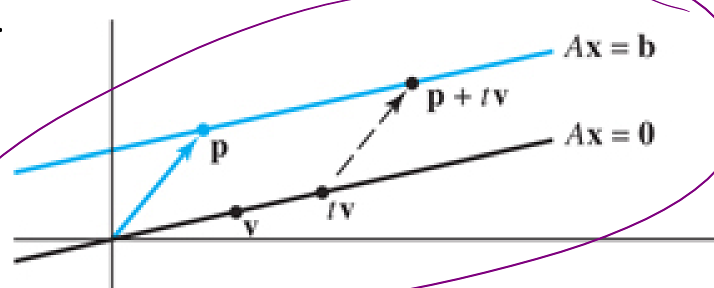
$x_2 = -2x_3 + 1$

$$\vec{x} = \begin{bmatrix} 5x_3 - 2 \\ -2x_3 + 1 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$t \quad \vec{v} \quad \vec{p}$



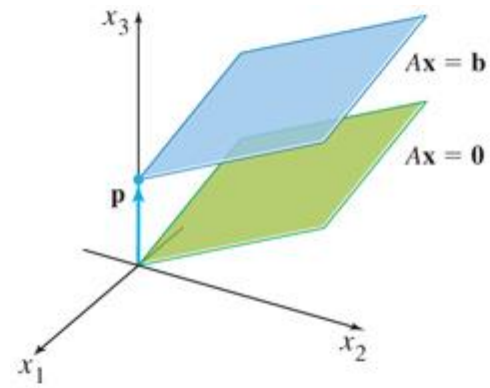
To visualize the solution set of $A\mathbf{x} = \mathbf{b}$ geometrically, we can think of vector addition as a translation.



The solution set of $A\mathbf{x} = \mathbf{b}$ is a line through $\mathbf{p}$ parallel to the solution set of $A\vec{x} = \vec{0}$.

THEOREM 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.



Prove the first part of Theorem 6: Suppose that $\mathbf{p}$ is a solution of $A\mathbf{x} = \mathbf{b}$, so that $A\mathbf{p} = \mathbf{b}$. Let $\mathbf{v}_h$ be any solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$, and let $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, Show that $\mathbf{w}$ is a solution to $A\mathbf{x} = \mathbf{b}$.

$A\vec{p} = \vec{b}$, $A\vec{v}_h = \vec{0}$, Show $\vec{w} = \vec{p} + \vec{v}_h$ solution $A\vec{x} = \vec{b}$

$$A\vec{w} = A(\vec{p} + \vec{v}_h) = A\vec{p} + A\vec{v}_h = \vec{b} + \vec{0} = \vec{b}$$

Q.E.D.

Writing a Solution Set (of a Consistent System) in Parametric Vector Form

1. Row reduce the augmented matrix to reduced echelon form.

2. Express each basic variable in terms of any free variables appearing in an equation.

pivot columns

3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.

4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

Ex 4: Each of the following equations determines a plane in $\mathbb{R}^3$. Do the two planes intersect? If so, describe their intersection.

$$x_1 + 4x_2 - 5x_3 = 0$$
$$2x_1 - x_2 + 8x_3 = 9$$

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} -2R_1 + R_2 \begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & -9 & 18 & 9 \end{bmatrix} -\tfrac{1}{9}R_2$$

$$\begin{bmatrix} 1 & 4 & -5 & 0 \\ 0 & 1 & -2 & -1 \end{bmatrix} -4R_2 + R_1 \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$x_1 = -3x_3 + 4$

$x_2 = 2x_3 - 1$

$$\vec{x} = \begin{bmatrix} -3x_3 + 4 \\ 2x_3 - 1 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$$

Ex 5: Write the general solution of $10x_1 - 3x_2 - 2x_3 = 7$ in parametric vector form,

$x_1 = .3x_2 + .2x_3 + .7$

$$\vec{x} = \begin{bmatrix} .3x_2 + .2x_3 + .7 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} .7 \\ 0 \\ 0 \end{bmatrix}$$

1.6 – Applications (read/re-54iew Network Flow as well – pages 53 – 54)

Balancing Chemical Equations

Chemical equations describe the quantities of substances consumed and produced by chemical reactions. For instance, when propane gas burns, the propane (C_3H_8) combines with oxygen (O_2) to form carbon dioxide (CO_2) and water (H_2O), according to an equation of the form

$$(x_1)C_3H_8 + (x_2)O_2 \rightarrow (x_3)CO_2 + (x_4)H_2O \qquad (4)$$

$$C_3H_8 : \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix}, O_2 : \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, CO_2 : \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, H_2O : \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \begin{matrix} \leftarrow \text{Carbon} \\ \leftarrow \text{Hydrogen} \\ \leftarrow \text{Oxygen} \end{matrix}$$

$$x_1 \begin{bmatrix} 3 \\ 8 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} - x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - x_4 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & -1 & 0 & 0 \\ 8 & 0 & 0 & -2 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & -5/4 & 0 \\ 0 & 0 & 1 & -3/4 & 0 \end{bmatrix}$$

$x_1 = 1/4\, x_4$

$x_2 = 5/4\, x_4$

$x_3 = 3/4\, x_4$

Let $x_4 = 4$

$$\boxed{C_3H_8 + 5\, O_2 = 3CO_2 + 4H_2O}$$