1.4 – Matrix Equations

Definition

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, **is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**; that is,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

$A\mathbf{x}$ is only defined if the number of ___columns___ of A equals the number of ___entries___ in $\mathbf{x}$.

$$\underset{m \times \cancel{n}}{A} \; \underset{\cancel{n} \times 1}{\vec{x}} = \underset{m \times 1}{\vec{b}}$$

Ex 1: (A is 2x3)

$$\begin{bmatrix} 2 & -1 & 5 \\ 0 & 3 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -1 \\ 3 \end{bmatrix} + (-3)\begin{bmatrix} 5 \\ -6 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + \begin{bmatrix} -4 \\ 12 \end{bmatrix} + \begin{bmatrix} -15 \\ 18 \end{bmatrix} = \begin{bmatrix} -15 \\ 30 \end{bmatrix} = \vec{b}$$

(A is 3x2)

$$\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - 1\begin{bmatrix} -4 \\ -5 \\ -6 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \\ 12 \end{bmatrix}$$

Ex 2: For $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^3$ Write the linear combination of $5\mathbf{u}_1 - \mathbf{u}_2 + 2\mathbf{u}_3$ as a matrix times a vector.

$$5\vec{u}_1 - \vec{u}_2 + 2\vec{u}_3 = \underbrace{\begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}}_{\substack{3\times 3 \\ A}} \underbrace{\begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}}_{\vec{x}}$$

Ex 3: Write the system of equations

$$\begin{aligned} 3x_1 - x_2 - 4x_3 &= 3 \\ x_1 \quad\;\; - 5x_3 &= -2 \end{aligned}$$

as a

a) Vector Equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$$

$$x_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -4 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad (5)$$

b) Matrix Equation

$$A \vec{x} = \vec{b}$$

$$\begin{bmatrix} 3 & -1 & -4 \\ 1 & 0 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad (4)$$

$$\begin{array}{cccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{b} \end{array}$$
$$\begin{bmatrix} 3 & -1 & -4 & 3 \\ 1 & 0 & -5 & -2 \end{bmatrix} \qquad (6)$$

Theorem 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and if $\mathbf{b}$ is in $\mathbb{R}^m$, the matrix equation

$$A\mathbf{x} = \mathbf{b} \qquad (4)$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \qquad (5)$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \qquad (6)$$

The equation $A\mathbf{x}=\mathbf{b}$ has a solutions if and only if **b** is a linear combination of the columns of *A*.

Ex 4: Let $A=\begin{bmatrix} 1 & -3 & -4 \\ -3 & 2 & 6 \\ 5 & -1 & -8 \end{bmatrix}$ and $\mathbf{b}=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x}=\mathbf{b}$ consistent for all possible b_1, b_2, b_3?

$$\begin{bmatrix} 1 & -3 & -4 & b_1 \\ -3 & 2 & 6 & b_2 \\ 5 & -1 & -8 & b_3 \end{bmatrix} \begin{matrix} 3R_1+R_2 \\ -5R_1+R_3 \end{matrix} \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1+b_2 \\ 0 & 14 & 12 & -5b_1+b_3 \end{bmatrix} 2R_2+R_3 \begin{bmatrix} 1 & -3 & -4 & b_1 \\ 0 & -7 & -6 & 3b_1+b_2 \\ 0 & 0 & 0 & b_1+2b_2+b_3 \end{bmatrix}$$

No, must satisfy $\boxed{b_1+2b_2+b_3=0}$

Ex: $b_1=1$
$b_2=-1$
$b_3=1$

Theorem 4

Let *A* be an $m \times n$ matrix. Then the following statements are logically equivalent. they are all true statements or they are all false.

a. For each **b** in $\mathbb{R}^m$, the equation $A\mathbf{x}=\mathbf{b}$ has a solution.

$\forall$ For every

b. Each **b** in $\mathbb{R}^m$ is a linear combination of the columns of *A*.

c. The columns of *A* span $\mathbb{R}^m$. $\rightarrow$ $\forall \vec{v} \in \mathbb{R}^m$, $\vec{v} = c_1\vec{a}_1 + c_2\vec{a}_2 + \dots c_n\vec{a}_n$

d. *A* has a pivot position in every row. $m \leq n$

$$\begin{bmatrix} * & * & & & & \\ & & * & 0 & & \\ & & & & * & \\ & & & & & * \end{bmatrix}$$

(Warning: *A* is a coefficient matrix here, not an augmented matrix.)

Ex 5: Compute $A\mathbf{x}=\mathbf{b}$ for $A=\begin{bmatrix} 1 & 4 & -1 \\ 2 & 0 & -3 \\ -3 & -2 & 5 \end{bmatrix}$ and $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$A\vec{x} = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 0 & -3 \\ -3 & -2 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ -3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + 4x_2 - x_3 \\ 2x_1 \quad\quad -3x_3 \\ -3x_1 - 2x_2 + 5x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \vec{b}$$

Row–Vector Rule for Computing $A\mathbf{x}$

If the product $A\mathbf{x}$ is defined, then the i th entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Ex 6: Compute

a) $\begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1(1) - 2(2) + 3(5) \\ 0(1) + 4(2) - 1(5) \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1(a) + 0(b) + 0(c) \\ 0(a) + 1(b) + 0(c) \\ 0(a) + 0(b) + 1(c) \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

(This is called the <u>Identity</u> matrix, denoted by I)

If I_n represents nxn identity matrix, then $I_n\mathbf{x}=\mathbf{x}$ for every $\mathbf{x}\in\mathbb{R}^n$

Theorem 5

If A is an $m \times n$ matrix, **u** and **v** are vectors in $\mathbb{R}^n$, and c is a scalar, then:

a. $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

b. $A(c\mathbf{u}) = c(A\mathbf{u})$.