1.3 – Vector Equations

Math 220
Warnock - Class Notes

Vectors in $\mathbb{R}^2$

A matrix with one column is called a column vector or vector

$\vec{u} = \mathbf{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ $\quad \mathbf{v} = \begin{bmatrix} .7 \\ \pi \\ 37/12 \end{bmatrix}$ $\quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ $\quad \begin{bmatrix} 2 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Vectors are equivalent if and only if the corresponding entries are equal.

The sum of the vectors $\mathbf{u}$ and $\mathbf{v}$ is the vector $\vec{u} + \vec{v} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$.

The scalar multiple of vector $\mathbf{u}$ by a real number *c* is the vector $c\mathbf{u}$ where each entry of $\mathbf{u}$ is multiplied by *c*. $\quad c\vec{u} = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$

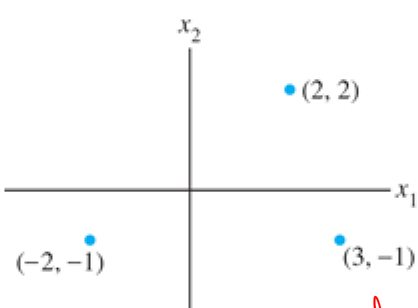
Ex 1: Given $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ find

a) $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 3 + (-1) \\ -2 + 4 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}$ $\quad (2, 2)$

b) $3\mathbf{u} = \begin{bmatrix} 3(3) \\ 3(-2) \end{bmatrix} = \boxed{\begin{bmatrix} 9 \\ -6 \end{bmatrix}}$

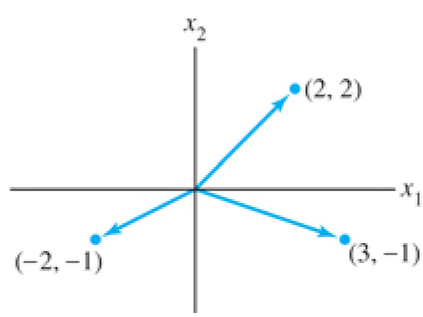
c) $2\mathbf{u} - 5\mathbf{v} \begin{bmatrix} 2(3) - 5(-1) \\ 2(-2) - 5(4) \end{bmatrix} = \boxed{\begin{bmatrix} 11 \\ -24 \end{bmatrix}}$

Geometric Descriptions of $\mathbb{R}^2$



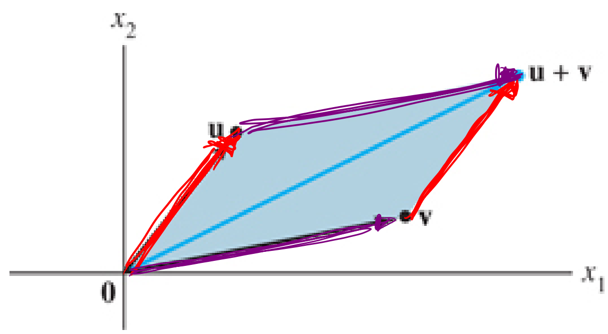
ordered pairs

vectors



Parallelogram Rule for Addition

If **u** and **v** in $\mathbb{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are **u**, **0**, and **v**. See Figure 3.

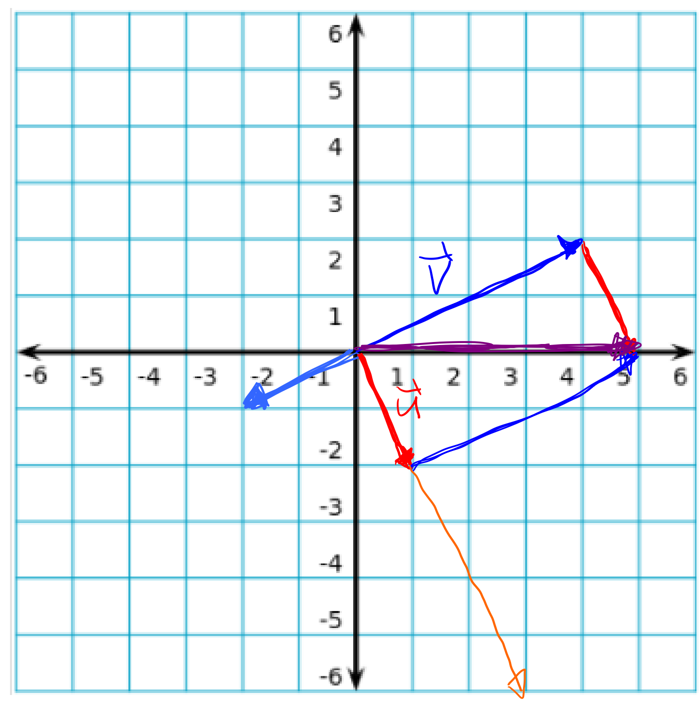


Ex 2: Given $\mathbf{u}=\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix}4\\2\end{bmatrix}$, draw their vectors and the following.

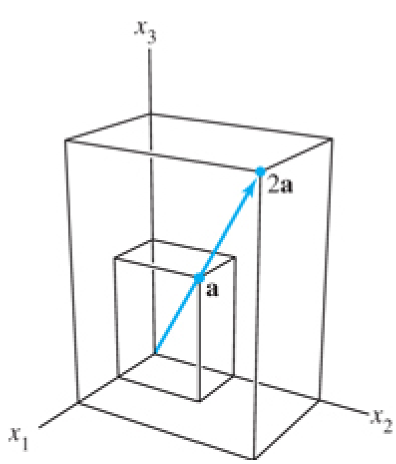
a) $\mathbf{u}+\mathbf{v} = \begin{bmatrix}1+4\\-2+2\end{bmatrix} = \begin{bmatrix}5\\0\end{bmatrix}$

b) $3\mathbf{u} = \begin{bmatrix}3(1)\\3(-2)\end{bmatrix} = \begin{bmatrix}3\\-6\end{bmatrix}$

c) $-\dfrac{1}{2}\mathbf{v} = \begin{bmatrix}-\frac{1}{2}(4)\\-\frac{1}{2}(2)\end{bmatrix} = \begin{bmatrix}-2\\-1\end{bmatrix}$



Vectors in $\mathbb{R}^3$



Vectors in $\mathbb{R}^n$

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}$$

The Zero vector has entries of all zero, denoted by $\mathbf{0}$ or

$$\vec{0}=\begin{bmatrix}0\\0\\0\\\vdots\\0\end{bmatrix}$$

Algebraic Properties of $\mathbb{R}^n$

For all **u**, **v**, **w** in $\mathbb{R}^n$ and all scalars c and d:

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ commutative prop

(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ assoc. prop.

(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$

(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$, where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$

(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(vii) $c(d\mathbf{u})=(cd)\mathbf{u}$

(viii) $1\mathbf{u}=\mathbf{u}$

Prove (i) and (v)

commutative property of real numbers

(i) $$\vec{u}+\vec{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}=\begin{bmatrix}v_1+u_1\\v_2+u_2\\\vdots\\v_n+u_n\end{bmatrix}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}+\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\vec{v}+\vec{u}$$

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Q.E.D.

(v) $$c(\vec{u}+\vec{v})=c\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}=\begin{bmatrix}c(u_1+v_1)\\c(u_2+v_2)\\\vdots\\c(u_n+v_n)\end{bmatrix}=\begin{bmatrix}cu_1+cv_1\\cu_2+cv_2\\\vdots\\cu_n+cv_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}+\begin{bmatrix}cv_1\\cv_2\\\vdots\\cv_n\end{bmatrix}$$

Distributive Property of Real numbers

$$=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+c\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$$

$$=c\vec{u}+c\vec{v}$$

Q.E.D.

Linear Combinations

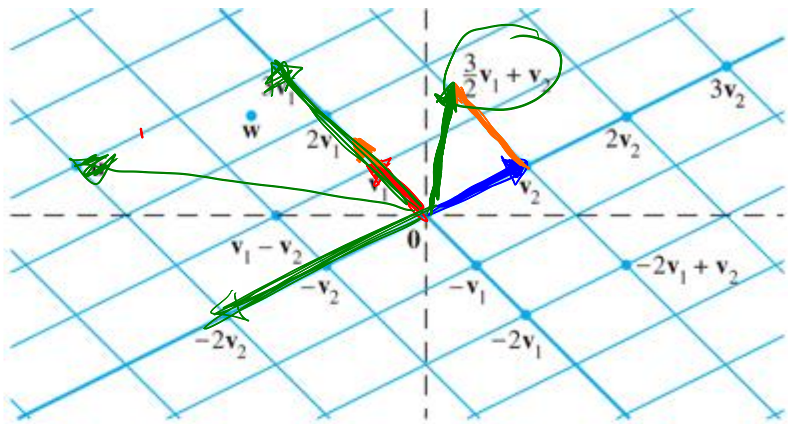
Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector **y** defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ with **weights** $c_1, \ldots, c_p$.

Ex 3: Figure 8 identifies selected linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$\vec{u} = 3\vec{v}_1 + (-2)\vec{v}_2$



Ex 4: Determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \qquad \begin{bmatrix} x_1 + 5x_3 \\ -2x_1 + x_2 - 6x_3 \\ +2x_2 + 8x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = 2 - 5x_3$

$x_2 = 3 - 4x_3$

x_3 is free

Let $x_3 = 0$

$$\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} \checkmark$$

$2\vec{a}_1 + 3\vec{a}_2 = \vec{b}$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n \quad \mathbf{b}] \qquad (5)$$

In particular, **b** can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

Definition

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by** $\mathbf{v}_1, \ldots, \mathbf{v}_p$. That is, Span $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

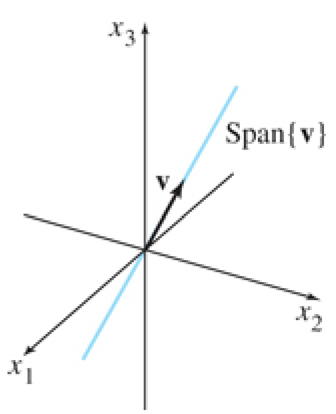
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

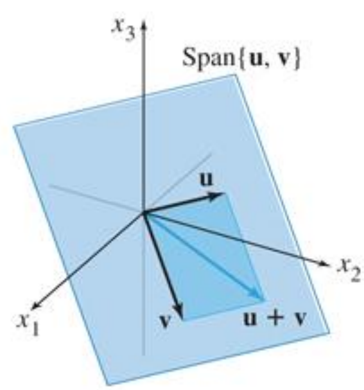
with $c_1, \ldots, c_p$ scalars.

$$\mathbf{b} \in \mathrm{Span}\left\{\mathbf{v}_1, \mathbf{v}_{2,} \ldots \mathbf{v}_p\right\}?$$

Every scalar multiple of individual vectors, $c\mathbf{v}_k$?

Geometric Description of Span $\{\mathbf{v}\}$ and Span $\{\mathbf{u}, \mathbf{v}\}$





Ex 5: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 10 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$. $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane in $\mathbb{R}^3$.

Is **b** in that plane?

Ex 6: Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 8 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -5 \\ -3 \end{bmatrix}$

For what value(s) of h is **y** in the plane generated by $\mathbf{v}_1$ and $\mathbf{v}_2$?