1.2 – Row Reduction & Echelon Forms

Math 220
Warnock - Class Notes

Definition

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros. (zero rows at the bottom)
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column. (all zeros above & below leading 1's)

The following matrices that we saw in section 1.1 are in

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Echelon Form   reduced echelon form

Ex 1: Here are matrices in

Echelon Form (no restrictions on the numbers)

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

Reduced Echelon Form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Nonzero matrices can be row-reduced into many different matrices in Echelon form. However, the Reduced Echelon Form of any matrix is unique – there is only one.

Theorem 1 Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix. unique

Definition

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A **pivot column** is a column of A that contains a pivot position.

Ex 2: Row reduce the matrix to echelon form, and locate the pivot columns.

$$\begin{bmatrix} -3 & 1 & -18 & -5 & 4 & 4 \\ 1 & 1 & 2 & 3 & 1 & 1 \\ -1 & 1 & -8 & -1 & 0 & 0 \\ 1 & 2 & -1 & 4 & -5 & -5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 1 \\ -3 & 1 & -18 & -5 & 4 & 4 \\ -1 & 1 & -8 & -1 & 0 & 0 \\ 1 & 2 & -1 & 4 & -5 & -5 \end{bmatrix} \begin{matrix} \\ 3R_1+R_2 \\ R_1+R_3 \\ -R_1+R_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 1 \\ 0 & 4 & -12 & 4 & 7 & 7 \\ 0 & 2 & -6 & 2 & 1 & 1 \\ 0 & 1 & -3 & 1 & -6 & -6 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_4} \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -3 & 1 & -6 & -6 \\ 0 & 2 & -6 & 2 & 1 & 1 \\ 0 & 4 & -12 & 4 & 7 & 7 \end{bmatrix} \begin{matrix} \\ \\ -2R_2+R_3 \\ -4R_2+R_4 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -3 & 1 & -6 & -6 \\ 0 & 0 & 0 & 0 & 13 & 13 \\ 0 & 0 & 0 & 0 & 31 & 31 \end{bmatrix}$$

$$\begin{matrix} \\ \\ \frac{1}{13}R_3 \\ \frac{1}{31}R_4 \end{matrix} \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -3 & 1 & -6 & -6 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{-R_3+R_4} \begin{bmatrix} 1 & 1 & 2 & 3 & 1 & 1 \\ 0 & 1 & -3 & 1 & -6 & -6 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot Positions

Pivot Collums

Ex 3: Use elementary row operations to transform the following matrix into echelon form and then reduced echelon form.

$$\begin{bmatrix} 2 & -4 & 3 & -4 & -11 & 28 \\ -1 & 2 & -1 & 2 & 5 & -13 \\ 0 & 0 & -3 & 1 & 6 & -10 \\ 3 & -6 & 10 & -8 & -28 & 61 \end{bmatrix}$$

Forward Phase vs. Backward Phase

Solutions of Linear Systems

Looking at the reduced echelon form of the matrix from Ex 3, we can describe our solution set to the corresponding system of equations to this augmented matrix.

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \end{array}$$
$$\begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{aligned} x_1 - 2x_2 \quad + 2x_5 &= 3 \\ x_3 \quad - x_5 &= 2 \\ x_4 + 3x_5 &= -4 \end{aligned}$$

$$\begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{cases} 2x_2 - 2x_5 + 3 \\ x_2 \text{ is free} \\ x_5 + 2 \\ -3x_5 - 4 \\ x_5 \text{ is free} \end{cases}$$

$$x_1 = 2x_2 - 2x_5 + 3$$
$$x_3 = x_5 + 2$$
$$x_4 = -3x_5 - 4$$

x_2, x_5 are free variables

The variables that are arbitrary, this text calls __free__ variables, and the others that rely on those __free__ variables or are fixed are called __basic__ variables.

Ex 4: Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 0 & -2 & 4 & 3 & 3 \\ 0 & 1 & 3 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} -3R_3 + R_1 \\ -2R_3 + R_2 \end{array} \begin{bmatrix} 1 & 0 & -2 & 4 & 0 & -3 \\ 0 & 1 & 3 & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} x_1 \quad -2x_3 + 4x_4 &= -3 \\ x_2 + 3x_3 - x_4 &= -3 \\ x_5 &= 2 \end{aligned}$$

$$x_1 = 2x_3 - 4x_4 - 3$$
$$x_2 = -3x_3 + x_4 - 3$$
$$x_5 = 2$$

x_3, x_4 are free

$$\begin{bmatrix} 2x_3 - 4x_4 - 3 \\ -3x_3 + x_4 - 3 \\ x_3 \\ x_4 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} -3 \\ -3 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -6 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

Theorem 2 Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$[0 \quad \cdots \quad 0 \quad b] \qquad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

Ex 5: Determine the existence and uniqueness of the linear systems represented by the augmented matrices that we've seen over the last two sections.

a) (1.1, #4)

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Exists? Yes!

Because No $[0\ 0\ 0\ 0\ 0\ b]$ row

Unique? Yes $\Rightarrow$ no free variables

$(5, 3, -1)$

b) (1.1, #5)

Does not Exist

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$0 \neq 2$

c) (1.2, ~~#3~~)

$$\begin{bmatrix} 1 & 0 & 5 & 2 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Exists

Not unique

X_3, X_4 are free variables

Using Row Reduction to Solve a Linear System

1. Write the augmented matrix of the system.

2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.

3. Continue row reduction to obtain the reduced echelon form.

4. Write the system of equations corresponding to the matrix obtained in step 3.

5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.