1.1 – Systems of Linear Equations

Math 220
Warnock - Class Notes

A linear equation of the variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$$

Where $b,\ a_1,\ a_2,\ \ldots,\ a_n$ are real or complex numbers.

Ex 1: Circle the linear equations, and state why the non-linear aren't linear.

a) $3x_1 - x_2 = 5x_1$

b) $4x_2 + 5 = \sqrt{x_1}$ ← radical
No

c) $x_1 - 4x_2 = x_1x_2$ ← product of variables
No

d) $-2x_1 + 7x_2 - \pi x_3 = \sqrt{2}$

A system of linear equations is a collection of one or more linear equations with the same variables. For example

$$\begin{aligned} 3x_1 - x_2 - 4x_3 &= 3 \\ x_1 \qquad - 5x_3 &= -2 \end{aligned}$$

A solution of a system is a list of numbers $(s_1, s_2, \ldots, s_n)$ that make every equation of the system true, when each s_k is substituted for x_k.

Ex 2: Verify that (3,2,1) is a solution to the system
$$\begin{aligned} 3x_1 - x_2 - 4x_3 &\overset{?}{=} 3 \\ x_1 \qquad - 5x_3 &= -2. \end{aligned}$$

$3(3) - 2 - 4(1) = 9 - 2 - 4$
$= 3$ ✓

$3 - 5(1) = -2$ ✓

The set of all possible solutions is called the solution set.

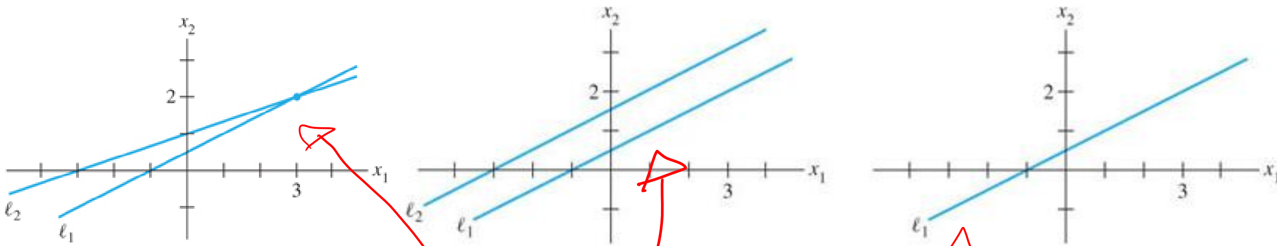
Ex 3: Find another solution to the system from Ex 2.

$(8, 13, 2)$

$(-2, -9, 0)$ ← $x_3 = 0$
$\Rightarrow x_1 = -2$
$\Rightarrow x_2 = -9$

Two systems are considered equivalent if they have the same solution set.

From 2 dimensional systems of equations in algebra, we should remember that there are 3 possibilities for the number of solutions to a system.



A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system is called consistent if it has at least one solution, and inconsistent if it has no solutions.

Matrix Notation

We will represent a system of equations by its coefficients in a matrix.

$$\begin{aligned} x_1 \qquad\quad -3x_3 &= 8 \\ 2x_1+2x_2+9x_3 &= 7 \\ x_2+5x_3 &= -2 \end{aligned}$$

will be re-written as the

coefficient matrix

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix} \quad 3\times 3$$

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{array}\right] \quad 3\times 4$$

The size of a matrix tells how many rows and collumns a matrix has.

An $m\times n$ matrix has

m rows and

n collumns.

Solving a Linear System – We are going to describe an algorithm for solving linear systems, which replaces one system with an equivalent one that is easier to solve. Since they are equivalent, they have the same solution set.

Ex 4: Solve the system

$$\begin{aligned} x_1 \quad\quad -3x_3 &= 8 \\ 2x_1+2x_2+9x_3 &= 7 \\ x_2+5x_3 &= -2 \end{aligned}$$

Three Operations we can use:

- swap rows
- scalar multiple of a row
- add a scalar multiple of one row to another and replace it.

$\boxed{-2R_1+R_2} \to R_2$

$$\begin{aligned} x_1 \quad -3x_3 &= 8 \\ +2x_2+15x_3 &= -9 \\ x_2+5x_3 &= -2 \end{aligned}$$

$\boxed{\frac{1}{2}R_2} \to R_2$

$$\begin{aligned} x_1 \quad -3x_3 &= 8 \\ x_2+\tfrac{15}{2}x_3 &= -\tfrac{9}{2} \\ x_2+5x_3 &= -2 \end{aligned}$$

$\boxed{-R_2+R_3} \to R_3$

$$\begin{aligned} x_1 \quad -3x_3 &= 8 \\ x_2+\tfrac{15}{2}x_3 &= -\tfrac{9}{2} \\ -\tfrac{5}{2}x_3 &= \tfrac{5}{2} \end{aligned}$$

$\boxed{-\frac{2}{5}R_3}$

$$\begin{aligned} x_1 \quad -3x_3 &= 8 \\ x_2+\tfrac{15}{2}x_3 &= -\tfrac{9}{2} \\ x_3 &= -1 \end{aligned}$$

$\boxed{x_3=-1}$

$$x_2+\tfrac{15}{2}(-1) = -\tfrac{9}{2}$$

$$+\tfrac{15}{2} \qquad +\tfrac{15}{2}$$

$\boxed{x_2=3}$

$$x_1 - 3(-1) = 8$$

$\boxed{x_1=5}$

$\boxed{(5,3,-1)}$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix}$$

$$\tfrac{1}{2}R_2 \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 15/2 & -9/2 \\ 0 & 1 & 5 & -2 \end{bmatrix} -R_2+R_3. \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 15/2 & -9/2 \\ 0 & 0 & -5/2 & 5/2 \end{bmatrix}$$

$$-\tfrac{2}{5}R_3. \begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 15/2 & -9/2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{-\frac{15}{2}R_3+R_2}$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{3R_3+R_1}$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Elementary Row Operations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

Two matrices are called row equivalent if there are a sequence of elementary row operations that transform one matrix into the other.

If two systems are row equivalent, they have the same solution set.

Two Fundamental Questions About a Linear System

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

Ex 5: Determine whether the systems are consistent or inconsistent. Do not solve.

$$\begin{aligned} x_2 + 4x_3 &= -5 \\ x_1 + 3x_2 + 5x_3 &= -2 \\ 3x_1 + 7x_2 + 7x_3 &= 6 \end{aligned}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} -3R_1 + R_3$$

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} 2R_2 + R_3$$

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$\boxed{\text{inconsistent}}$

$0x_1 + 0x_2 + 0x_3 = 2$

$$\begin{aligned} x_1 \qquad + 3x_3 \qquad &= 2 \\ x_2 \qquad - 3x_4 &= 3 \\ -2x_2 + 3x_3 + 2x_4 &= 1 \\ 3x_1 \qquad + 7x_4 &= -5 \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix} -3R_1 + R_4$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} 2R_2 + R_3$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} 3R_3 + R_4$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 3 & -4 & 7 \\ 0 & 0 & 0 & -5 & 10 \end{bmatrix}$$

consistent