4.1 – Preliminary Theory – Linear Equations

An n[th]-order linear Initial Value Problem has the form

$$Solve: \quad a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = g(x)$$

$$Subject\,to: \; y(x_0) = y_0, \; y'(x_0) = y_1, \; \ldots, \; y^{(n-1)}(x_0) = y_{n-1}$$

The solution $y(x)$ **must**

a) **pass through** (x_0, y_0)

b) **have slope** y_1 **at** (x_0, y_0)

c) **have concavity** y_2 **at** (x_0, y_0)

etc.

> **THEOREM 4.1.1 Existence of a Unique Solution**
>
> Let $a_n(x)$, $a_{n-1}(x)$, . . . , $a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

$6e^0 - 2e^0 - 3$

$y = 3e^{2x} + e^{-2x} - 3x$ **is a solution to** $y'' - 4y = 12x$**,** $y(0) = 4$**,** $y'(0) = 1$.

We know that it's unique because $a_2(x) = 1 \neq 0$ **on any interval containing** $x = 0$.

$y = cx^2 + x + 3$ **is a solution to** $x^2y'' - 2xy' + 2y = 6$**,** $y(0) = 3$**,** $y'(0) = 1$ **for ANY c.**

This is not unique, because $a_2(x) = x^2 = 0$ at $x = 0$.

A 2nd-order boundary-value problem (BVP) has the form

$$\text{Solve:} \quad a_2(x)y''+a_1(x)y'+a_0(x)y=g(x)$$

$$\text{Subject to:} \quad y(a)=y_0,\ y(b)=y_1$$

These are called boundary conditions.

There many possible pairs of boundary conditions.

$$y'(a)=y_0, \qquad y(b)=y_1$$

$$y(a)=y_0, \qquad y'(b)=y_1$$

$$y'(a)=y_0, \qquad y'(b)=y_1,$$

A BVP could have one solution, no solutions, or many solutions.

Ex #1. **Given that $y=c_1\sin(6x)+c_2\cos(6x)$ is a 2-parameter family of solutions to the DE $y''+36y=0$, find the solutions with the boundary conditions:**

a) $y(0)=0,\ y\left(\frac{\pi}{2}\right)=0$

$0=c_1\cdot\sin 0+c_2\cos 0 \Rightarrow 0=c_2$

$0=c_1\sin 3\pi+c_2\cos 3\pi \Rightarrow 0=-c_2$

c_1 is arbitrary

$\boxed{y=c_1\sin 6x}$ (infinite solutions)

b) $y(0)=0,\ y\left(\frac{\pi}{2}\right)=1$

$c_2=0$

$1=c_1\sin 3\pi+c_2\cos 3\pi$

$1=-c_2 \Rightarrow c_2=-1$

Contradiction $\rightarrow$ $\boxed{\text{no solution}}$

c) $y(0)=0,\ y\left(\frac{\pi}{4}\right)=1$

$1=c_1\sin\frac{3\pi}{2}+c_2\cos\frac{3\pi}{2}$

$c_2=0$

$1=c_1\cdot(-1)$

$c_1=-1$

$\boxed{y=-\sin 6x}$

A ___homogeneous___ DE has the form

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_0(x)y=0$$

Whereas, a ___nonhomogeneous___ DE has the form

$$\underbrace{a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+...+a_0(x)y}_{L(y)}=g(x)$$

To solve a non-homogeneous DE, we will first solve the associated homogeneous DE.

Assumptions: (for simplicity)

a) $a_i(x),\ i=0,1,2,...,n$ **and** $g(x)$ **are continuous**

b) $a_n(x)\neq 0$ **for every x in the interval**

D is called a ___differential___ ___operator___.

$$D(y)=y'=\frac{dy}{dx}$$

$D(D(y))=D(y')=y''$ **- We can also say** $D^2(y)=y''$ **and** $D^n(y)=y^{(n)}$

An nth-order Differential Operator is

$$L=a_n(x)D^n+...+a_1(x)D+a_0(x)$$
$$L(y)=a_n(x)y^{(n)}+...+a_1(x)y'+a_0(x)y$$

L **is linear, so**

$$L(y_1+y_2)=L(y_1)+L(y_2)$$
$$L(cy)=cL(y)$$

Therefore, *y* is a solution to the homogeneous DE if

$$L(y) = a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = 0$$

$L(y) = 0$

Theorem 4.1.2: **Superposition Principle – Homogeneous Equations**

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_0(x)y = 0$$

on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \ldots + c_ky_k(x)$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

solutions of $L(y)=0$

Proof:

Let $y = c_1y_1(x) + c_2y_2(x) + \ldots + c_ky_k(x)$ (c_i are arbitrary constants)

$L(y) = L(c_1y_1(x) + c_2y_2(x) + \ldots + c_ky_k(x))$

$= c_1L(y_1(x)) + c_2L(y_2(x)) + \ldots + c_kL(y_k(x))$

Lemma - comes before

$= c_1\cdot 0 + c_2\cdot 0 + \ldots + c_k\cdot 0$ ($y_1 - y_k$ are solutions of $L(y)=0$)

$= 0$ $\therefore$ y is a solution to $L(y)=0$

COROLLARIES TO THEOREM 4.1.2

(A) A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.

(B) A homogeneous linear differential equation always possesses the trivial solution $y = 0$.

$\boxed{\textbf{Ex \#1.}}$ $y_1 = x^2$ **and** $y_2 = x^2 \ln x$ **are both solutions to** $x^3y''' - 2xy' + 4y = 0$ **on the interval** $(0, \infty)$**. By the superposition principle,**

$$y = c_1x^2 + c_2x^2\ln x$$

is also a solution.

$y_1 = x^2$

$y_1' = 2x$

$y_1'' = 2$

$y''' = 0$

$-2x\cdot 2x + 4x^2 = 0$ ✓

$y_2' = 2x\ln x + x$

$y_2'' = 2\ln x + 2 + 1$

$y_2''' = \frac{2}{x}$

$x^3\cdot\frac{2}{x} - 2x(2x\ln x + x) + 4(x^2\ln x)$

$\cancel{2x^2} - \cancel{4x^2\ln x} - \cancel{2x^2} + \cancel{4x^2\ln x} = 0$ ✓

DEFINITION 4.1.1 Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent.**

In other words, a set of functions is linearly independent if the ONLY constants for

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

are $c_1 = c_2 = \ldots = c_n = 0$

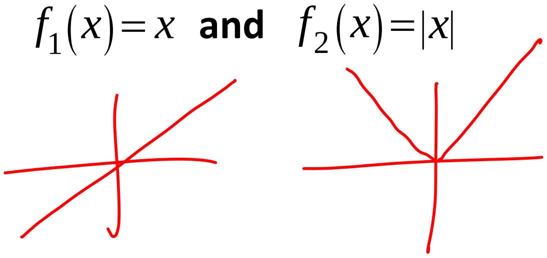
Linear dependence for two functions simply means one can be written as a constant multiple of the other.

Examples:

$f_1(x) = \sin(2x)$ **and** $f_2(x) = \sin x \cos x$

$$\left.\begin{aligned} f_1(x) &= \sin(2x) = 2\sin x\cos x \\ &= 2 f_2(x) \\ f_1(x) - 2f_2(x) &= 0 \end{aligned}\right\} \quad f_1 \,\&\, f_2 \text{ are linearly dependent}$$

$f_1(x) = x$ **and** $f_2(x) = |x|$



$c_1 x + c_2 |x| = 0$

$c_1 = c_2 = 0$ is the only solution

Ex #2. **Show** $y_1 = x$ and $y_2 = x^2$ **are linearly independent on (-2,2).**

Suppose $c_1 x + c_2 x^2 = 0$

If $x=1$ $\quad c_1 + c_2 = 0$

$c_1 = -c_2$

If $x=-1$ $\quad -c_1 + c_2 = 0$

$c_2 = c_1$

$c_2 = -c_2$

$2c_2 = 0$

$\boxed{c_2 = 0 = c_1}$

$\therefore$ linearly independent

Ex #3. **Show that** $y_1 = \cos^2 x,\ y_2 = \sin^2 x,\ y_3 = \sec^2 x,\ y_4 = \tan^2 x$ **are linearly dependent on** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$**.**

$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$

$\tan^2 x + 1 = \sec^2 x$

$\tan^2 x - \sec^2 x = -1$

$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0$

Let

$c_1 = 1$

$c_2 = 1$

$c_3 = -1$

$c_4 = 1$

$\underbrace{\cos^2 x + \sin^2 x}_{1} \underbrace{- \sec^2 x + \tan^2 x}_{-1}$

$= 0$

$\therefore$ linearly dependent

DEFINITION 4.1.2 Wronskian

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$\begin{vmatrix} {}^{+}a & {}^{-}b & {}^{+}c \\ {}^{-}d & {}^{+}e & {}^{-}f \\ {}^{+}g & {}^{-}h & {}^{+}i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_0(x)y=0 \quad \textbf{(6)}$$

THEOREM 4.1.3 Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

So, when $y_1, y_2, \ldots, y_n$ are solutions on I, $W(y_1, y_2, \ldots, y_n)$ is either

- **identically zero (linearly dependent solution), or**
- **never zero on the interval (linearly independent solution)**

DEFINITION 4.1.3 Fundamental Set of Solutions

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order differential equation (6) on an interval I is said to be a **fundamental set of solutions** on the interval.

THEOREM 4.1.4 Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogeneous linear nth-order differential equation (6) on an interval I.

THEOREM 4.1.5 General Solution—Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x),$$

where c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Ex #4. **Use the Wronskian to determine whether the following solutions are linearly independent or linearly dependent. Assume they are solutions.**

a) $y_1 = \cos(2x)$, $y_2 = \sin(2x)$ **for** $y'' + 4y = 0$.

$$\begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) - (-2\sin^2(2x))$$
$$= 2(\cos^2(2x) + \sin^2(2x))$$
$$= 2 \cdot 1 = 2 \neq 0$$

$\therefore$ linearly independent

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

$$e^{2x} e^{-x} = e^x$$

b) $y_1 = e^x$, $y_2 = e^{2x}$, $y_3 = e^{-x}$ **for** $y''' - 2y'' - y' + 2y = 0$.

$$\begin{vmatrix} {}^{+}e^x & e^{2x} & e^{-x} \\ {}^{-}e^x & 2e^{2x} & -e^{-x} \\ {}^{+}e^x & 4e^{2x} & e^{-x} \end{vmatrix} = e^x \begin{vmatrix} 2e^{2x} & -e^{-x} \\ 4e^{2x} & e^{-x} \end{vmatrix} - e^x \begin{vmatrix} e^{2x} & e^{-x} \\ 4e^{2x} & e^{-x} \end{vmatrix} + e^x \begin{vmatrix} e^{2x} & e^{-x} \\ 2e^{2x} & -e^{-x} \end{vmatrix}$$

$$e^x\left[\cancel{2e^x} + 4e^x - e^x + 4e^x + -e^x - \cancel{2e^x}\right]$$

$$e^x \cdot 6e^x = 6e^{2x} \neq 0 \quad \therefore \text{linearly independent}$$

c) $y_1 = 5$, $y_2 = \cos^2 x$, $y_3 = \sin^2 x$

$$\begin{vmatrix} 5 & \cos^2 x & \sin^2 x \\ 0 & -2\cos x \sin x & 2\sin x \cos x \\ 0 & -2\cos 2x & 2\cos 2x \end{vmatrix} = 5 \cdot \begin{vmatrix} -\sin 2x & \sin 2x \\ -2\cos 2x & 2\cos 2x \end{vmatrix} + 0 \left| \quad \right| + 0 \left| \quad \right|$$

$(-\sin 2x)$ $\quad$ $(\sin 2x)$

$-\sin(2x)$

$$= 5(\underbrace{-2\sin 2x \cos 2x + 2\sin 2x \cos 2x}_{0})$$

$$= 5 \cdot 0 = 0 \quad \therefore \text{linearly dependent}$$

$3x+7y>2$

$$a_n(x)y^{(n)}+a_{n-1}(x)y^{(n-1)}+\ldots+a_0(x)y=g(x) \qquad \textbf{(7)}$$

THEOREM 4.1.6 General Solution—Nonhomogeneous Equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (7) on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (6) on I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p,$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

y_p **here is a particular solution free of arbitrary parameters.**

For example, $y_p = 3$ **is a particular solution of** $y''+9y=27$

Proof: Let $Y(x)$ & $y_p(x)$ be any particular solution of (7) $L(y)=g(x)$

Call $u(x) = Y(x) - y_p(x)$

$L(u) = L(Y - y_p) = L(Y) - L(y_p) = g(x) - g(x) = 0$

u is a solution to the homogeneous $L(y)=0$

$u = c_1y_1 + c_2y_2 + \ldots + c_ny_n$

$$\boxed{Y(x) = u(x) + y_p(x) = c_1y_1 + c_2y_2 + \ldots + c_ny_n + y_p}$$

So the general solution of a nonhomogeneous linear equation consists of two functions: $y = \underbrace{c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)} + y_p(x) = y_c(x) + y_p(x).$

$y_c(x)$ **is called the** complimentary Function.

THEOREM 4.1.7 Superposition Principle—Nonhomogeneous Equations

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation (7) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x), \tag{12}$$

where $i = 1, 2, \ldots, k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x) \tag{13}$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y$$
$$= g_1(x) + g_2(x) + \cdots + g_k(x). \tag{14}$$

35. (a) Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$ are, respectively, particular solutions of *(time-permitting)*

$$y'' - 6y' + 5y = -9e^{2x}$$

and $$y'' - 6y' + 5y = 5x^2 + 3x - 16.$$

(b) Use part (a) to find particular solutions of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$$

$$y_p = x^2 + 3x + 3e^{2x}$$

and $$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}.$$

$$y'' - 6y' + 5y = -2(5x^2 + 3x - 16) - \frac{1}{9}(-9e^{2x})$$

$$y_p = -2(x^2 + 3x) - \frac{1}{9}(3e^{2x})$$

$$y_p = -2x^2 - 6x - \frac{1}{3}e^{2x}$$