13.3: Arclength and Curvature

Saturday, October 15, 2022 2:55 PM

Arc Length and Curvature

❖ **Length of a curve**

We defined the length of a parametric curve in two dimensions as $L=\int_{\alpha}^{\beta}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ provided that the various functions and parameters are all well-behaved.

The ~~arc~~ length in three dimensions is defined similarly: Suppose $\vec{r}(t)=\langle f(t),g(t),h(t)\rangle$ on $a\le t\le b$ where f', g' and h' are continuous. If the curve is traveled exactly once as t increases from a to b, then:

$$\boxed{\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\,dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt \end{aligned}}$$

Notice that we can also write: $L=\int_a^b |\vec{r}\,'(t)|\,dt$

Example 1: A glider is soaring upward along the helix $\vec{r}(t)=3\sin t\,\vec{i}+3\cos t\,\vec{j}+4t\,\vec{k}$. How long is the glider's path from $t=0$ to $t=2\pi$? 2π

we need $L=\int_0^{2\pi}|\vec{r}\,'(t)|\,dt$

13.3 ex. Helix.

① $\vec{r}\,'(t)=\langle 3\cos t, -3\sin t, 4\rangle$

② $|\vec{r}\,'(t)|=\sqrt{9\cos^2 t+9\sin^2 t+16}=\sqrt{25}=5$

③ $L=\int_0^{2\pi}5\,dt = 2\pi(5)=10\pi$

❖ Arc Length

The length of a curve is a constant (a number). The arc length is similar, but a function.

$$\text{Length of curve} \quad L = \int_a^b |\vec{r}\,'(t)|\,dt \quad \leftarrow \quad \text{constant}$$

$$\text{Arc length} \quad s(t) = \int_a^t |\vec{r}\,'(u)|\,du \quad \leftarrow \quad \text{function}$$

dummy variable u.

Notice that $\frac{ds}{dt} = |\vec{r}\,'(t)|$

❖ Curvature

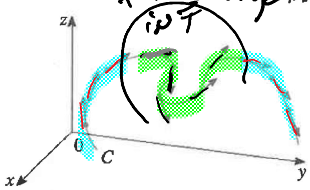
A parametrization $\vec{r}(t)$ is called **smooth** on an interval I if $\vec{r}\,'(t)$ is continuous and non-zero on I. A curve is called **smooth** if it has a smooth parameterization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

If C is a smooth curve defined by the vector function $\vec{r}(t)$, recall that the unit tangent vector $\vec{T}(t)$ is given by $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ and indicates the direction of the curve.

memorize

s(t).

In the picture, you can see that $\vec{T}(t)$ changes directions very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists sharply.



fast change in $\vec{T}$

manipulate 14.29

The curvature κ at a given point is a measure of how quickly the curve changes direction at a point.

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| \quad \text{(curvature definition \#1)}$$

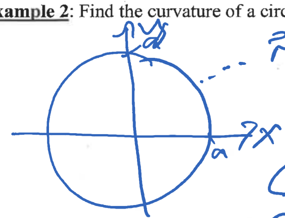
hard to work w/

manipulate 14.30

While possible, it is a nuisance to find $\vec{T}$ as a function of s. But notice that with the chain rule, we have $\frac{d\vec{T}}{ds}\cdot\frac{ds}{dt} = \frac{d\vec{T}}{dt}$. We can solve for $\frac{d\vec{T}}{ds}$ and $\frac{d\vec{T}}{ds} = \dfrac{\frac{d\vec{T}}{dt} \;\leftarrow\; \vec{T}'}{\frac{ds}{dt} \;\leftarrow\; |\vec{r}\,'(t)|}$ and thus:

$$\kappa = \left|\frac{\vec{T}\,'(t)}{\vec{r}\,'(t)}\right| \quad \text{(curvature definition \#2)}$$

Example 2: Find the curvature of a circle with radius a, centered at the origin, on the xy-plane.



$\vec{r}(t) = \langle a\cos t, a\sin t \rangle$ parameterize

goal: find $k = \left| \frac{\vec{T}'(t)}{\vec{r}'(t)} \right|$

① $\vec{r}'(t) = \langle -a\sin t, a\cos t \rangle$

② $|\vec{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$

③ $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{1}{a}\langle -a\sin t, a\cos t \rangle$
$= \langle -\sin t, \cos t \rangle$

big circle
⇒ small curvature.

small circle
⇒ big curvature

④ $\vec{T}'(t) = \langle -\cos t, -\sin t \rangle$

⑤ $k = \left| \frac{\vec{T}'(t)}{\vec{r}'(t)} \right| = \left| \left\langle -\frac{\cos t}{a}, -\frac{\sin t}{a} \right\rangle \right|$
$= \sqrt{\frac{\cos^2 t}{a^2} + \frac{\sin^2 t}{a^2}} = \frac{1}{a}$

Although the previous formulas work for finding the curvature, the following formula is more convenient to apply.

memorize me.

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3} \quad \text{(curvature definition \#3)}$$

Note: This formula is not intuitive and its derivation requires (1.) using the product rule, (2.) the fact that $\vec{T} \times \vec{T} = \vec{0}$, and (3.) knowing that $|\vec{T}(t)| = 1$ implies $\vec{T} \cdot \vec{T}\,' = 0$. The full derivation is in the text.

Example 3: Find the curvature of a straight line parametrized by $\vec{r}(t) = \vec{C} + t\vec{v}$.

need: $\vec{r}'(t) = \vec{v}$ (constant vector since a line)

$\vec{r}''(t) = \vec{0}$

$|\vec{r}'(t)|$

$\vec{r}' \times \vec{r}'' = \vec{v} \times \vec{0} = \vec{0}$

$k = \frac{|\vec{0}|}{|\vec{r}'(t)|^3} = 0$ ← not 0

straight line has curvature 0.

Example 4: Find the curvature of $\vec{r}(t)=\langle t^2, 2t+1, e^t\rangle$ at the point $(0,1,1)$.

$t=0$

① $\vec{r}\,'(t)=\langle 2t, 2, e^t\rangle$

② $\vec{r}\,''(t)=\langle 2, 0, e^t\rangle$

All derivatives are taken, now we can evaluate.

(1*) $\vec{r}\,'(0)=\langle 0, 2, 1\rangle$ check w/ dot product

(2*) $\vec{r}\,''(0)=\langle 2, 0, 1\rangle$

(3) $\vec{r}\,'(0)\times\vec{r}\,''(0)=\langle 2, -(-2), -4\rangle=\langle 2, 2, -4\rangle$

(3*) $|\vec{r}\,'(0)\times\vec{r}\,''(0)|=\sqrt{4+4+16}=\sqrt{24}$

(1**) $|\vec{r}\,'(0)|=\sqrt{0+4+1}=\sqrt{5}$

conclusion $k=\dfrac{\sqrt{24}}{(\sqrt{5})^3}$

❖ **The Normal and Binormal Vectors**

At a given point on a smooth space curve $\vec{r}(t)$, there are many vectors that are ~~orthogonal to~~ the unit tangent vector $\vec{T}(t)$. We'll show that $\vec{T}\,'(t)$ is one of them!

① $\vec{T}(t)\cdot\vec{T}(t)=|\vec{T}(t)|^2=1$.

② $\dfrac{d}{dt}(1)=0=\dfrac{d}{dt}\left[\vec{T}(t)\cdot\vec{T}(t)\right]=\vec{T}\,'(t)\cdot\vec{T}(t)+\vec{T}(t)\cdot\vec{T}\,'(t)$

$\Rightarrow 0=2\,\vec{T}(t)\cdot\vec{T}\,'(t)$ $\quad\therefore \boxed{\vec{T}\perp\vec{T}\,'}$

Historical note: The mathematicians Jean Frederic Frenet (1847) and Joseph Alfred Serret (1851) independently discovered and described the kinematic properties of a particle moving along a curve using the tangent, normal, and binormal vectors. However, our modern notation for vectors and linear algebra did not exist for them. Today, their formulas are called the Frenet-Serret formulas and relate what we now call the TNB-frame and the curvature κ and torsion τ.

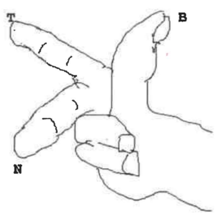
Since $\vec{T}'(t)$ is not a generally a unit vector, we will define the **principal unit normal vector** $\vec{N}(t)$ (or simply **unit normal**) as: $\vec{N}(t)=\dfrac{\vec{T}'(t)}{|\vec{T}'(t)|}$. Note that $\vec{T}$ gives the direction of motion and $\vec{N}$ points in the direction the curve is turning. The vector $\vec{B}(t)=\vec{T}(t)\times\vec{N}(t)$ is called the **binormal vector** and it is perpendicular to $\vec{T}$ and $\vec{N}$.

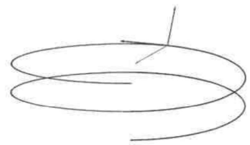
Manipulate 14.37

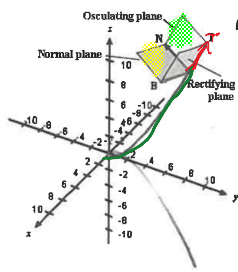
The plane determined by the normal and binormal vectors $\vec{N}$ and $\vec{B}$ at a point P on a curve C is called the **normal plane** of C at P. It consists of all lines that are orthogonal to the tangent vector $\vec{T}$.

The plane determined by the vectors $\vec{T}$ and $\vec{N}$ is called the **osculating plane** of C at P. The name comes from the Latin *osculum*, meaning "kiss." It is the plane that comes closest to containing the part of the curve near P.

Manipulate 14.38



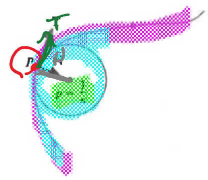




Note: For a plane curve, the osculating plane is simply the plane that contains the curve.

in 2D

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which $\vec{N}$ points), and has radius $\rho=\dfrac{1}{\kappa}$ and is called the **osculating circle** of C at P. It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.



Manipulate 14.5 ex. 70

Nontechnical description: Imagine a car moving along a curved road on a vast flat plane. Suddenly, at one point along the road, the steering wheel locks in its present position. Thereafter, the car moves in a circle that "kisses" the road at the point of locking. The curvature of the circle is equal to that of the road at that point. That circle is the osculating circle of the road curve at that point.

at $\langle 0, 3, 0\rangle$ when $t=0$

Example 5: Consider the space curve $\vec{r}(t) = 3\sin t\vec{i} + 3\cos t\vec{j} + 4t\vec{k}$ from example 1

The symphony of the strange language lulled her into a deep dreamless sleep.

13.03 ex TNB

a) Find the unit normal vector

① $\vec{r}'(t) = \langle 3\cos t, -3\sin t, 4\rangle$

② $|\vec{r}'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$

$= \sqrt{25} = 5$

③ $\vec{T}(t) = \langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5}\rangle$

④ $\vec{T}'(t) = \langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0\rangle$

DOT

⑤ $|\vec{T}'(t)| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t + 0} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

⑥ $\vec{N}(t) = \frac{1}{\frac{3}{5}}\langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0\rangle$

$= \langle -\sin t, -\cos t, 0\rangle$

check $\vec{T}\cdot\vec{N} = 0$

b) Find the binormal vector at $(0,3,0)$ when $t=0$

$\vec{B} = \vec{T}\times\vec{N} = \langle \frac{3}{5}, 0, \frac{4}{5}\rangle \times \langle 0, -1, 0\rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3/5 & 0 & 4/5 \\ 0 & -1 & 0 \end{vmatrix} = \langle \frac{4}{5}, -0, -\frac{3}{5}\rangle$$

check w/ dot product

$\vec{B} = \langle \frac{4}{5}, 0, -\frac{3}{5}\rangle$.

c) equation of the normal plane at $(0,3,0)$

plane that has normal vector $\vec{T} = \langle \frac{3}{5}, 0, \frac{4}{5} \rangle$

Normal plane:

$$\frac{3}{5}(x-0) + 0(y-3) + \frac{4}{5}(z-0) = 0$$

$$\Rightarrow \frac{3}{5}x + \frac{4}{5}z = 0.$$

d) equation of the osculating plane at $(0,3,0)$

plane w/ perpendicular vector $\vec{B} = \langle \frac{4}{5}, 0, -\frac{3}{5} \rangle$

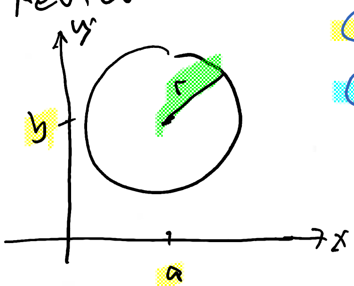
osculating plane: $\frac{4}{5}(x-0) + 0(y-3) - \frac{3}{5}(z-0) = 0$

$$\Rightarrow \frac{4}{5}x - \frac{3}{5}z = 0.$$

e) equation of the osculating circle at $(0,3,0)$

13.3 ex kissing circle

Review



① shared point $(0,3,0)$

② center of the circle

(a) radius $\rho = \frac{1}{\kappa}$

(b) $\langle 0,3,0 \rangle + \rho \vec{N}$

$= \langle 0,3,0 \rangle + \frac{25}{3} \langle 0,-1,0 \rangle$

$= \langle 0, -\frac{16}{3}, 0 \rangle$

recall) $\langle 0, -\frac{3}{5}, 0 \rangle$

$$\kappa = \left. \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} \right|_{t=0} = \frac{3/5}{5} = 3/25$$

$$\Rightarrow \rho = \frac{25}{3}$$

$\vec{r}(t) = \langle a + r\cos t, b + r\sin t \rangle$

$= \langle a, b \rangle + r(\cos t \langle 1, 0 \rangle + \sin t \langle 0, 1 \rangle)$

$\langle 0, -\frac{16}{3}, 0 \rangle$ $\quad \frac{25}{3}$ $\quad \vec{T}$ $\quad \vec{N}$

kissing circle: $\vec{r}(t) = \langle 0, -\frac{16}{3}, 0 \rangle + \frac{25}{3}\left(\cos t \langle \frac{3}{5}, 0, \frac{4}{5} \rangle + \sin t \langle 0, -1, 0 \rangle\right)$