12.5: Equations of Lines and Planes

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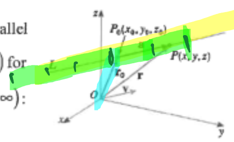
Equations of Lines and Planes

In this section we will learn how to use scalar and vector products to write equations for lines, line segments and planes in space. We will use these representations in chapter 13 and Calculus IV where the basic ideas will come up repeatedly.

❖ **Lines and Line Segments in Space**

In 2D, a line is determined by a point and the slope. In 3D, a line is determined by a point and the direction of the line which is described by a vector.

Suppose L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\vec{v}$. Then for some scalar **parameter** $t \in (-\infty, \infty)$:



$$\overrightarrow{P_0P} = t\vec{v}$$

Note that the value of t depends on the location of the point P along the line.

$$\langle x - x_0, \quad y - y_0, \quad z - z_0 \rangle = t\langle a, b, c \rangle$$

$$(x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k} = t(a\vec{i} + b\vec{j} + c\vec{k})$$

$$x\vec{i} - x_0\vec{i} + y\vec{j} - y_0\vec{j} + z\vec{k} - z_0\vec{k} =$$

$$x\vec{i} + y\vec{j} + z\vec{k} - (x_0\vec{i} + y_0\vec{j} + z_0\vec{k}) =$$

$$\underbrace{x\vec{i} + y\vec{j} + z\vec{k}}_{\vec{r}} = \underbrace{x_0\vec{i} + y_0\vec{j} + z_0\vec{k}}_{\vec{r}_0} + \underbrace{t(a\vec{i} + b\vec{j} + c\vec{k})}_{t\vec{v}}$$

t parameter
↑
variable

If $\vec{r}$ is the position vector of the arbitrary point $P(x, y, z)$ and $\vec{r}_0$ is the position vector of the point $P_0(x_0, y_0, z_0)$, then the vector equation of L is:

entrance ramp to line.
how far to go
direction

memorize this

$$\boxed{\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}}$$

This can be written as:

$$x\vec{i} + y\vec{j} + z\vec{k} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k} + ta\vec{i} + tb\vec{j} + tc\vec{k}$$

$$x\vec{i} + y\vec{j} + z\vec{k} = (x_0 + ta)\vec{i} + (y_0 + tb)\vec{j} + (z_0 + tc)\vec{k}$$

$$\langle x, y, z \rangle = \langle x_0 + ta, \quad y_0 + tb, \quad z_0 + tc \rangle$$

Manipulates
12.66
and
13.69

Hence:

$$\boxed{x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct}$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$. Each value of the parameter t gives a point (x, y, z) on L.

In general, if a vector $\mathbf{v} = \langle a, b, c\rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L. ⟵ similar to slope.

Example 1: Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\vec{v} = 2\vec{i} + 4\vec{j} - 2\vec{k}$. And come up with two other points on this line.

$\vec{r} = \langle -2, 0, 4\rangle + t\langle 2, 4, -2\rangle$

$x = -2 + 2t$
$y = 4t$
$z = 4 - 2t$

points on the line
$t = -2 \Rightarrow (-6, -8, 8)$
$t = 3 \Rightarrow (4, 12, -2)$

* Another way of describing a line L is to eliminate the parameter t from the parametric equations. If none of a, b, and c is 0, we can solve each these equations for t, equate the results, and obtain:

$$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

These equations are called **symmetric equations** of L. Notice that the numbers a, b, and c that appear in the denominators are the directions numbers of L, that is, components of a vector parallel to L. If one of a, b, or c is 0, we can still eliminate t. For instance if $a = 0$, we could write the equations of L as:

$$x = x_0 \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This means that L lies in the vertical plane $x = x_0$.

Two lines in 3D are called **skew lines** if they don't intersect and are not parallel!

Example 2: Decide if the following lines are parallel, they intersect or are skew.

$$L_1: \frac{x}{1} = \frac{y-1}{-1} = \frac{z-2}{3} \quad , \quad L_2: \frac{x-2}{2} = \frac{y-3}{-2} = \frac{z}{7}$$

Not scalar multiples
$\Rightarrow$ not parallel lines

$\frac{x}{1} = \frac{x-2}{2}$
$\Rightarrow 2x = x - 2$
$\Rightarrow x = -2$

$\frac{y-1}{-1} = \frac{y-3}{-2}$
$\Rightarrow -2y + 2 = -y + 3$
$\Rightarrow -1 = y$

$\frac{-2}{1} \overset{?}{\neq} \frac{-1-1}{-1} = 2$

These are skew lines

Example 3: Suppose we have two points $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

a) Find parametric equations for the line passing through them.

$\vec{r} = \vec{r}_0 + t\vec{v}$

$= \langle -3, 2, -3\rangle + t\langle 4, -3, 7\rangle$

$\vec{r}_0 = \langle -3, 2, -3\rangle$

$\vec{v} = \vec{PQ} = \langle 4, -3, 7\rangle$

$x = -3 + 4t$

$y = 2 - 3t$

$z = -3 + 7t$

NOTE: Parametrizations are not unique!

$x = 1 + 4t$

$y = -1 - 3t$

$z = 4 + 7t$

b) Find symmetric equations of the line.

$x \quad y \quad z$

$$t = \frac{x+3}{4} = \frac{y-2}{-3} = \frac{z+3}{7}$$

Manipulate 13.67 shows a similar projection as well as 13.68

c) At what point does this line intersects the yz-plane?

$x = 0$

$$\frac{0+3}{4} = \frac{y-2}{-3} = \frac{z+3}{7}$$

y:

$$\frac{3}{4} = \frac{y-2}{-3}$$

$$-\frac{9}{4} = y - 2$$

$$-\frac{1}{4} = y$$

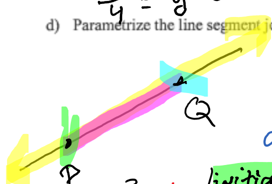
$$\frac{3}{4} = \frac{z+3}{7}$$

$$\Rightarrow \frac{21}{4} = z+3$$

$$\Rightarrow \frac{9}{4} = z$$

Intersect the plane @ $\left(0, -\frac{1}{4}, \frac{9}{4}\right)$.

d) Parametrize the line segment joining the two points.



$P(-3, 2, -3)$ and $Q(1, -1, 4)$ $\}$ Line

$\vec{r} = (1-t)\begin{pmatrix}\text{initial}\\ \text{position}\end{pmatrix} + t\begin{pmatrix}\text{final}\\ \text{position}\end{pmatrix}$

when $0 \le t \le 1$

$\vec{r}(t) = (1-t)\langle -3, 2, -3\rangle + t\langle 1, -1, 4\rangle$

when $0 \le t \le 1$

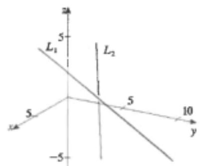
$t = 0: \vec{r} = 1\begin{pmatrix}\text{initial}\\ \text{position}\end{pmatrix}$

$t = 1: \vec{r} = 1\begin{pmatrix}\text{final}\\ \text{position}\end{pmatrix}$

Suppose we want to find the equation of the line segment connecting the two points P_0 (the "initial" point with position vector $\vec{r_0}$) and P_1 (the "end" point with position vector $\vec{r_1}$). Let $\vec{v} = \vec{r_1} - \vec{r_0}$ then:

$$\vec{r} = \vec{r_0} + t\vec{v} = \vec{r_0} + t\left(\vec{r_1} - \vec{r_0}\right) = (1-t)\vec{r_0} + t\vec{r_1}$$

Where $\vec{r}$ is the position vector for an arbitrary point P between P_0 and P_1.



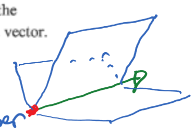
The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \leqslant t \leqslant 1$$

❖ **Equation for a Planes**

To start, let's talk about two interesting relationships between a vector and a plane.

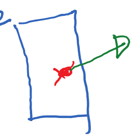
1) A vector is parallel to a plane if it lies on the plane, or else has no points in common with the plane. The latter happens when all the lines on the plane are either skew or parallel to that vector.
2) A vector is perpendicular to a plane if it is orthogonal to all the vectors on the plane.



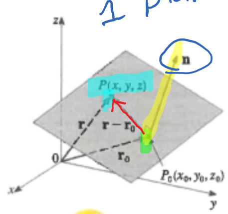
Suppose you have a point and a vector.

1) How many planes exist that are parallel to the vector and include the point? ∞ number

2) How many planes exist that are perpendicular to the vector and include the point? 1 plane



So a plane is determined by knowing a point $P_0(x_0, y_0, z_0)$ on the plane and its "tilt" or orientation. This "tilt" is defined by a vector that is orthogonal to the plane. This orthogonal vector $\vec{n}$ is called a **normal vector**.



Let $P(x, y, z)$ be an arbitrary point on the plane that contains $P_0(x_0, y_0, z_0)$. Then $\vec{n}$ is perpendicular to $\overrightarrow{P_0P}$. That is:

$$\vec{n} \cdot \overrightarrow{P_0P} = 0$$

This is called the **vector equation** of the plane.

manipulate
13.72

Don't memorize
yet.

If $\vec{n} = \langle a,b,c \rangle$ we can write:

vector that lives to the plane

$$\langle a,b,c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$$
$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

This is called the **scalar equation** of the plane.

Memorize

We can change the form of this equation by distributing a, b and c. So:

$$ax-ax_0+by-by_0+cz-cz_0=0$$
$$ax+by+cz-(ax_0+by_0+cz_0)=0$$

variables number

Note that $ax_0+by_0+cz_0$ is just a number.

So we can define the **linear equation** of the plane: $ax+by+cz+d=0$ where $d=-(ax_0+by_0+cz_0)$

NOTE: Generally we give our answers in either the form $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ which highlights the normal vector and a point or in the form $ax+by+cz=D$ from which we again see the normal vector and can easily find the three intercepts (assuming they all exist).

manipulate
13,73

Let's consider another form of writing the vector equation for the plane. If $\vec{r} = \langle x,y,z\rangle$ is the position vector of $P(x,y,z)$ and $\vec{r_0} = \langle x_0,y_0,z_0\rangle$ is the position vector of $P_0(x_0,y_0,z_0)$ then $\overrightarrow{P_0P} = \vec{r} - \vec{r_0}$, hence:

$$\boxed{\mathbf{n}\cdot(\mathbf{r}-\mathbf{r}_0)=0}$$

don't memorize

which can also be written as:

$$\boxed{\mathbf{n}\cdot\mathbf{r}=\mathbf{n}\cdot\mathbf{r}_0}$$

Historical Note: There are a lot of formulas in this section which makes it easy to lose track of what you are doing. At the end of the day, you are just learning about linear equations. This isn't new; rather you've been exploring linear equations since pre-algebra (1 variable as in $2x+3=4$) and then in algebra (2 variables as in $2x+3y=4$) and later systems of equations (such as $2x+3y=4; 5x+6y=7$). This section just expands linear into three dimensions!

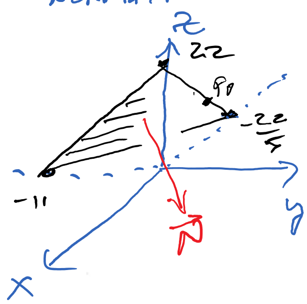
In studying linear equations, you are carrying on an old tradition that spans at least Africa (Egypt), the middle-East (Babylon), Asia (China), and eventually even Europe where they caught on once they adopted algebra from the Arabs and the number system used in India.

If this sounds cool, then you are going to love linear algebra where you spend a full term studying systems of linear equations including their many applications.

Example 4: Find an equation for the plane through $P_0(-3,0,7)$ perpendicular to $\vec{n} = 5\vec{i} + 2\vec{j} - \vec{k}$.
Then find the intercepts and sketch the plane!

point

normal.



plane

$$5(x-(-3)) + 2(y-0) - 1(z-7) = 0$$

$$\Rightarrow 5x + 15 + 2y - z + 7 = 0$$

$$\Rightarrow 5x + 2y - z = -22$$

$y = z = 0$	$x = z = 0$	$x = y = 0$
$x = -\frac{22}{5}$	$y = -11$	$z = 22$

Example 5: Find an equation for the plane through $P(2,-1,3)$, $Q(1,4,0)$, and $R(0,-1,5)$

$$\left.\begin{aligned} \vec{PQ} &= \langle -1, 5, -3 \rangle \\ \vec{PR} &= \langle -2, 0, 2 \rangle \end{aligned}\right\} \text{vectors on the plane.}$$

manipulate
13.75

$$\vec{PQ} \times \vec{PR} = \vec{n} \quad \text{normal vector}$$

$$\vec{n} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 5 & -3 \\ -2 & 0 & 2 \end{vmatrix}$$

$$= \langle 10, +8, 10 \rangle$$

plane

$$10(x-2) + 8(y+1) + 10(z-3) = 0$$

$-(-1)$

❖ **Intersection of Planes with Lines and other Planes**

Example 6: What is the relationship between the line $x=\frac{8}{3}+2t \quad y=-2t \quad z=1+t$ and the plane $3x+2y+6z=6$? How would you know if they are parallel? If they intersect find the intersection point(s).

parallel to $\langle 2,-2,1\rangle$

$\vec{N}=\langle 3,2,6\rangle$

Test for parallel: $\langle 3,2,6\rangle \cdot \langle 2,-2,1\rangle$
$= 6-4+6 \neq 0$
the line is not parallel to the plane

Does the line intersect the plane?

solve $3\left(\frac{8}{3}+2t\right)+2(-2t)+6(1+t)=6$

$\Rightarrow 8+6t-4t+6+6t=6$
$\Rightarrow 8t=-8$
$\Rightarrow t=-1.$ ← parameter

Find point
$x=\frac{8}{3}+2(-1)$
$=2/3$
$y=2$
$z=0$

The line and plane intersect at
$\left(\frac{2}{3},2,0\right)$

Relationships between two planes can be described as:

1) Two planes are parallel if and only if their normal vectors are parallel. That is $\vec{n_1}=k\vec{n_2}$ for some scalar k.
2) Two planes that are not parallel intersect in a line. Note that the line of intersection is perpendicular to both planes' normal vectors. That is parallel to the cross product of the two normal.
3) The angle between the two planes is defined as the acute angle between their normal vectors.

Example 7: Suppose we have two planes: $P_1: 3x-6y-2z=15$ and $P_2: 2x+y-2z=5$.

a) Find a vector parallel to the line of intersection of the planes.

$\vec{N}_1 = \langle 3, -6, -2 \rangle$

$\vec{N}_2 = \langle 2, 1, -2 \rangle$

$\vec{N}_1 \times \vec{N}_2 = \langle 14, 4, 15 \rangle \longleftarrow$ vector parallel to the line of intersection

b) Find parametric equations for the intersection line.

point and direction

We just need 1 point of intersection! suppose $x=0$

$\cancel{3x} - 6y - 2z = 15$
$\cancel{2x} + y - 2z = 5$
$\Longrightarrow y = -\frac{10}{7}$, $z = -\frac{45}{14}$

point $\left(0, -\frac{10}{7}, -\frac{45}{14}\right)$ lies on both planes and on the line.

line: $\vec{r} = \left\langle 0, -\frac{10}{7}, -\frac{45}{14} \right\rangle + t\langle 14, 4, 15 \rangle$

c) Find symmetric equations for the intersection line.

$$\frac{x}{14} = \frac{y + 10/7}{4} = \frac{z + 45/14}{15}$$

$x = 14t$

$y = -\frac{10}{7} + 4t$

$z = -\frac{45}{14} + 15t$

d) Find the angle between the two planes. = Angle between normal vectors

$\vec{P}_1 = \langle 3, -6, -2 \rangle$

$\vec{P}_2 = \langle 2, 1, -2 \rangle$

recall $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$

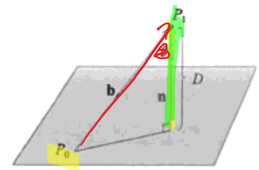
$\Rightarrow \theta = \cos^{-1} \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{7 \cdot 3}\right)$

$= \cos^{-1}\left(\frac{4}{21}\right)$

❖ Distance between Points, Planes and Lines

To find the distance D of the **point** $P_1(x_1, y_1, z_1)$ **to the plane** $ax+by+cz+d=0$, pick a point on the plane, call it $P_0(x_0, y_0, z_0)$. Then the vector connecting P_0 to P_1 can be found as $\vec{b}=\overrightarrow{P_0P}=\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle$. We know the equation of the plane so we know its normal vector $\vec{n}=\langle a,b,c\rangle$. Using the given picture, we can think about distance D in two different ways:



1) Considering the right triangle, then $\cos\theta=\dfrac{D}{|\vec{b}|}$. We also know that θ is the angle between $\vec{n}$ and $\vec{b}$ so $\cos\theta=\dfrac{\vec{n}\cdot\vec{b}}{|\vec{n}||\vec{b}|}$. Putting them equal to each other, $\dfrac{D}{|\vec{b}|}=\dfrac{\vec{n}\cdot\vec{b}}{|\vec{n}||\vec{b}|}$ and multiplying both sides by $|\vec{b}|$ we get $D=\dfrac{\vec{n}\cdot\vec{b}}{|\vec{n}|}$ but the dot product can be positive or negative and distance is always positive, hence we need to take the absolute value of the dot product: $D=\dfrac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}$

2) Considering the magnitude of the projection (component) of $\overrightarrow{P_0P}$ onto the normal, that is:
$$D=\left|\text{comp}_{\vec{n}}\vec{b}\right|=\frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}$$

They are equal! So now we can find the formula for distance:

$$D=\frac{|\vec{n}\cdot\vec{b}|}{|\vec{n}|}=\frac{|\langle a,b,c\rangle\cdot\langle x_1-x_0, y_1-y_0, z_1-z_0\rangle|}{|\vec{n}|}=\frac{|a(x_1-x_0)+b(y_1-y_0)+b(z_1-z_0)|}{|\vec{n}|}$$

$$=\frac{|ax_1-ax_0+by_1-by_0+cz_1-cz_0|}{|\vec{n}|}=\frac{|ax_1+by_1+cz_1-(ax_0+by_0+cz_0)|}{|\vec{n}|}$$

$$=\frac{|ax_1+by_1+cz_1-(ax_0+by_0+cz_0)|}{|\vec{n}|}$$

knowing $d=-(ax_0+by_0+cz_0)$, this formula can be written succinctly as: $D=\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$

know it exists, but don't memorize.

Example 8: Find the distance from $P_1(1,1,3)$ to the plane $3x+2y+6z=6$.

$$D=\frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$$=\frac{|3(1)+2(1)+6(3)-6|}{\sqrt{9+4+36}}$$

$$=\frac{17}{7}$$ distance from P_1 to the plane.

To find the distance between **two parallel planes**, we find any points on one plane and calculate its distance to the other plane. We do the same to find the distance **between a line and a plane** (where the line and plane are parallel).

Example 9: Find the distance between the parallel planes $P_1: x+2y+6z=1$ and $P_2: x+2y+6z=10$.

$(1,0,0)$ is a point on P_1

find distance from $(1,0,0)$ to plane P_2.

$$D=\frac{|1(1)+2(0)+6(0)-10|}{\sqrt{1+4+36}}$$

$$=\frac{9}{\sqrt{41}}$$ distance between the planes.

To find the distance between **two skew or parallel lines**, view the lines as lying on two parallel planes and then proceed as above. (Note that we have to find the equation of a plane that includes one of the lines and is parallel to the other.)

Example 10: Find the distance between the skew lines $L_1: \frac{x}{1}=\frac{y-1}{-1}=\frac{z-2}{3}$, $L_2: \frac{x-2}{2}=\frac{y-3}{-2}=\frac{z}{7}$

$\vec{v}_1=\langle 1,-1,3\rangle$

$\vec{v}_2=\langle 2,-2,7\rangle$

$\vec{v}_1\times\vec{v}_2=\langle -1,-1,0\rangle$

point from L_1: $(0,1,2)$

Together these form a plane

$-(x-0)-(y-1)+0(z-2)=0$

$\Rightarrow -x-y=-1$

- lines parallel to plane
- L_1 on the plane

Distance from the plane to L_2

$(2,3,0)$ is on L_2

$$D=\frac{|-1(2)-1(3)+0(0)+1|}{\sqrt{1+1}}=\frac{4}{\sqrt{2}}$$