12.3: Dot Product

Wednesday, September 28, 2022 3:50 PM

The Dot Product and its Use!

❖ **Dot Product**

Unlike numbers, there are two ways to multiply vectors. We learn about dot product here and cross product in the next section.

> $\boxed{1}$ **Definition** If $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, then the **dot product of a and b** is the number $\mathbf{a}\cdot\mathbf{b}$ given by
>
> $$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

NOTE: The result of a dot product of two vectors is a scalar (number)! This is important.

Example 1: Find the following.

a) $\langle 1,-2,-1\rangle\cdot\langle -6,2,-3\rangle = 1\cdot(-6) + (-2)\cdot 2 + (-1)(-3)$

b) $\left(\frac{1}{2}\vec{i}+3\vec{j}+\vec{k}\right)\cdot\left(4\vec{i}-\vec{j}+2\vec{k}\right) = \frac{1}{2}(4) + 3(-1) + 1(2)$

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

> **Properties of the Dot Product** If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then
>
> 1. $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2$
> 2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$
> 3. $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{c}$
> 4. $(c\mathbf{a})\cdot\mathbf{b} = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$
> 5. $\mathbf{0}\cdot\mathbf{a} = 0$

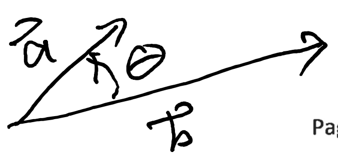
0 vector

0 scalar

We can easily prove these properties (some are in the book).

① $\langle a_1, a_2, a_3\rangle\cdot\langle a_1, a_2, a_3\rangle = a_1^2 + a_2^2 + a_3^2$

There is an equivalent geometric definition that physicists actually use as their starting place. You can show that these are equivalent using the Law of Cosines.



$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

relates vectors and angles.

3 Theorem If θ is the angle between the vectors **a** and **b**, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

manipulate
13-44

Note that the last Theorem gives you:

6 Corollary If θ is the angle between the nonzero vectors **a** and **b**, then

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example 2: Find the angle between $\vec{a} = \vec{i} - 2\vec{j} - 2\vec{k}$ and $\vec{b} = 6\vec{i} + 3\vec{j} + 2\vec{k}$.

$$\vec{a} \cdot \vec{b} = 6 + (-6) + -4 = -4$$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

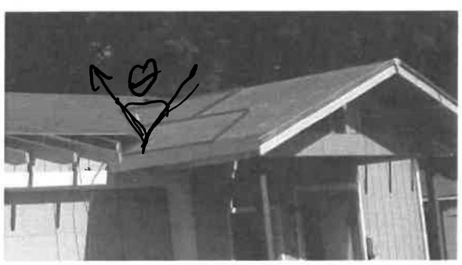
$$|\vec{b}| = \sqrt{36+9+4} = 7$$

$$\Rightarrow \cos\theta = \frac{-4}{3(7)}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{4}{21}\right)$$

$$\approx 1.76 \text{ rad.}$$

Teacher story: Remember the application about the roof … well one of the authors learned this the hard way! He was sheeting a new roof (putting new wood down under the roofing material) and needed to cut some angled pieces. Being a "smart" mathematician, he used his trig skills, it was close, but not quite right! Out in the real world, he couldn't figure out the angle!



So he did what most people would do … he used a lot of extra nails and just covered it up.

Later, he told an engineering friend about this who said, "Why didn't you use vectors and the dot product to find the angle?"

Lesson learned.

Example 3: Suppose one plane of a roof has a 6:12 pitch and it is intersected by a second plane that has a pitch of 4:12. Find the angle between the valley and a line going straight up the 6:12 roof plane.

Need 2 vectors.

① straight up roof

$\vec{u} = \langle 12, 0, 6 \rangle$

② up the valley

$\vec{v} = \langle 12, 18, 6 \rangle$

$\frac{4}{12} = \frac{6}{y}$

$\vec{u} \cdot \vec{v} = 144 + 0 + 36 = 180$

$|\vec{u}| = \sqrt{144 + 0 + 36} \approx 13.42$

$|\vec{v}| = \sqrt{144 + 324 + 36} \approx 22.45$

$\Rightarrow \theta \approx \cos^{-1}\left(\frac{180}{13.42 \cdot 22.45}\right)$

$\approx 53.31°$

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are called perpendicular or orthogonal if the angle between them is $\theta = \frac{\pi}{2}$. (Note: This is $90°$, however we focus on radians as that is the prevalent measure used throughout mathematics). Now we can calculate the dot product of the perpendicular vectors:

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\underbrace{\cos\left(\frac{\pi}{2}\right)}_{0} = 0$$

and conversely if $\vec{a} \cdot \vec{b} = 0$, then $\cos\theta = 0$, so $\theta = \frac{\pi}{2}$. The zero vector $\vec{0}$ is considered to be perpendicular to all vectors. Therefore we have the following method for determining whether two vectors are orthogonal.

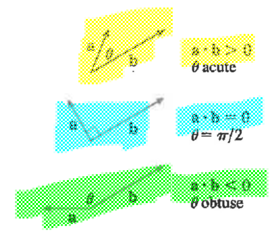
7	Two vectors **a** and **b** are orthogonal if and only if **a** · **b** = 0.

Example 4: Show that $\underset{\vec{a}}{\langle 3, -2, 1 \rangle}$ and $\underset{\vec{b}}{\langle 0, 2, 4 \rangle}$ are orthogonal.

$\vec{a} \cdot \vec{b} = 0 + (-4) + 4 = 0$

since the dot product is zero, the vectors are $\perp$.

Because $\cos\theta > 0$ if $0 \le \theta < \pi/2$ and $\cos\theta < 0$ if $\pi/2 < \theta \le \pi$, we see that $\mathbf{a}\cdot\mathbf{b}$ is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of $\mathbf{a}\cdot\mathbf{b}$ as measuring the extent to which $\mathbf{a}$ and $\mathbf{b}$ point in the same direction. The dot product $\mathbf{a}\cdot\mathbf{b}$ is positive if $\mathbf{a}$ and $\mathbf{b}$ point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2). In the extreme case where $\mathbf{a}$ and $\mathbf{b}$ point in exactly the same direction, we have $\theta = 0$, so $\cos\theta = 1$ and



$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|$$

If $\mathbf{a}$ and $\mathbf{b}$ point in exactly opposite directions, then $\theta = \pi$ and so $\cos\theta = -1$ and $\mathbf{a}\cdot\mathbf{b} = -|\mathbf{a}||\mathbf{b}|$.

❖ Direction Angles

The *direction angles* of a nonzero vector $\vec{a} = \langle a_1, a_2, a_3\rangle$ are the angles in the interval $[0,\pi]$, that the vector makes with the positive *x*-, *y*- and *z*-axis. The cosines of these angles are called the *direction cosines*.

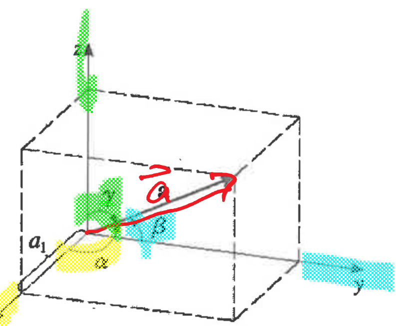
NOTE: This is a topic may need for homework, but direction angles are not used heavily elsewhere.

Looking at the given figure:

$$\cos\alpha = \frac{\vec{a}\cdot\vec{i}}{|\vec{a}||\vec{i}|} = \frac{\langle a_1, a_2, a_3\rangle\cdot\langle 1,0,0\rangle}{|\vec{a}|} = \frac{a_1}{|\vec{a}|}$$

$$\cos\beta = \frac{\vec{a}\cdot\vec{j}}{|\vec{a}||\vec{j}|} = \frac{\langle a_1, a_2, a_3\rangle\cdot\langle 0,1,0\rangle}{|\vec{a}|} = \frac{a_2}{|\vec{a}|}$$

$$\cos\gamma = \frac{\vec{a}\cdot\vec{k}}{|\vec{a}||\vec{k}|} = \frac{\langle a_1, a_2, a_3\rangle\cdot\langle 0,0,1\rangle}{|\vec{a}|} = \frac{a_3}{|\vec{a}|}$$



This gives us: $a_1 = |\vec{a}|\cos\alpha$, $a_2 = |\vec{a}|\cos\beta$, $a_3 = |\vec{a}|\cos\gamma$

So: $\vec{a} = \langle a_1, a_2, a_3\rangle = \langle |\vec{a}|\cos\alpha, |\vec{a}|\cos\beta, |\vec{a}|\cos\gamma\rangle$

Which can be written as: $\vec{a} = |\vec{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$

length of unit vec = 1

Therefore: Unit vector $\boxed{\dfrac{\vec{a}}{|\vec{a}|} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle}$

Also, we can see : $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{a_1^2}{|\vec{a}|^2} + \dfrac{a_2^2}{|\vec{a}|^2} + \dfrac{a_3^2}{|\vec{a}|^2} = \dfrac{a_1^2+a_2^2+a_3^2}{|\vec{a}|^2} = \dfrac{a_1^2+a_2^2+a_3^2}{\left(\sqrt{a_1^2+a_2^2+a_3^2}\right)^2} = 1$

Example 5: Find the direction angles of the vector $\vec{a} = 2\vec{i} + \vec{j} + 4\vec{k}$ (Round to the nearest whole angle).

unit vec $= \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{21}}\langle 2,1,4\rangle = \langle \cos\alpha, \cos\beta, \cos\gamma\rangle$

$\alpha = \cos^{-1}\left(\frac{2}{\sqrt{21}}\right) \approx 1.12$ rad. $\gamma = \cos^{-1}\left(\frac{4}{\sqrt{21}}\right) \approx 0.51$ rad.

$\beta = \cos^{-1}\left(\frac{1}{\sqrt{21}}\right) \approx 1.35$ rad.

❖ **Projections**

Think about vector $\vec{v} = \langle a, b\rangle$ in 2D.

We say that a is the horizontal component of vector $\vec{v}$. We can also say it is the component of vector $\vec{v}$ in the direction of x-axis (onto $\vec{i}$ or any vector in that direction). We can write this as: $a = \text{comp}_{\vec{i}}\vec{v}$.
scalar

Also we call $a\vec{i}$ the shadow of $\vec{v}$ onto x-axis. In correct mathematical terms, $a\vec{i}$ is the projection of $\vec{v}$ onto $\vec{i}$ and can be written as: $a\vec{i} = \text{proj}_{\vec{i}}\vec{v}$
vector

We can define component or projection of a vector onto any other vector or in any other direction.

The **vector projection** of $\vec{b}$ onto $\vec{a}$ is denoted by $\text{proj}_{\vec{a}}\vec{b}$.

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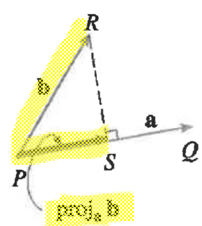
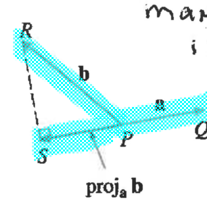
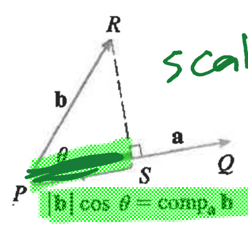
The two cases are shown in the figure.





The **scalar projection** of $\vec{b}$ onto $\vec{a}$ (also called the **component** of $\vec{b}$ along $\vec{a}$) is denoted by $\text{comp}_{\vec{a}}\vec{b}$. It is defined to be the signed magnitude of the vector projection $\left(|\vec{b}|\cos\theta\right)$.

* understand



scalar

Derivation: $\text{comp}_{\vec{a}}\vec{b} = |\vec{b}|\cos\theta = |\vec{b}|\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

Maybe memorize.

No arrows in the picture

Scalar projection of b onto a:	$\text{comp}_{\mathbf{a}}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert}$	scalar
Vector projection of b onto a:	$\text{proj}_{\mathbf{a}}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert}\right)\dfrac{\mathbf{a}}{\lvert\mathbf{a}\rvert} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert^2}\,\mathbf{a}$	vector

Notice that the vector projection is the scalar projection multiplied by the unit vector in the direction of vector $\bar{a}$

Example 6: Find the scalar and vector projection of $\vec{a} = 6\vec{i} + 3\vec{j} + 2\vec{k}$ onto $\vec{b} = \vec{i} - 2\vec{j} - 2\vec{k}$.

$$\text{comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{6-6-4}{\sqrt{36+4+4}} = \frac{-4}{7} \quad \text{scalar}$$

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{-4}{7}\left(\begin{matrix}\text{unit}\\ \text{vec} \parallel\\ \text{to } \vec{b}\end{matrix}\right) = -\frac{4}{7}\left\langle \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right\rangle \quad \text{vector}$$

$$\frac{1}{\sqrt{9}}\langle 1, -2, -2\rangle$$

❖ Work

The work done by a constant force $\vec{F}$ that moves an object from P to Q (creating displacement vector $\vec{D}$) can be calculated by:

work = force * distance

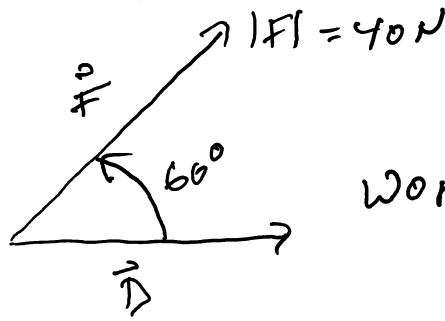
$$W = \vec{F} \cdot \vec{D}$$

↑ dot, not *

NOTE: Force and work are major themes in the last part of Calculus IV.

manipulate 13.53

Example 7: If $|\vec{F}| = 40N$, $|\vec{D}| = 3m$ and $\theta = 60°$, find the work!



$|\vec{D}| = 3M$

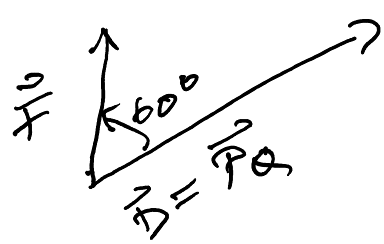
$$\begin{aligned} Work &= \vec{F} \cdot \vec{D} \\ &= |\vec{F}||\vec{D}|\cos\theta \\ &= 40(3)\underbrace{\cos 60°}_{\frac{1}{2}} \\ &= 60\ N \cdot M \end{aligned}$$

Example 8: How much work is done if a man pulls a wagon from point $P(-1,5)$ to point $Q(8,2)$ by applying 20 lbs. on the handle that makes a $60°$ angle with the horizon? (Displacement is in feet)

$\vec{PQ} = \langle 9, -3 \rangle$

and

$|\vec{PQ}| = \sqrt{81+9} = \sqrt{90}$



$|\vec{F}| = 20 lbs$

$$\begin{aligned} Work = \vec{F} \cdot \vec{D} &= 20 \cdot \sqrt{90} \cos 60° \\ &= 10\sqrt{90}\ ft \cdot lbs \end{aligned}$$

of work