10.2: Calculus with Parametric Curves

Tuesday, October 11, 2022 7:48 PM

Calculus with Parametric Curves

Having seen how to represent curves by parametric equations, we now apply the methods of calculus to these parametric curves. In particular, we solve problems involving tangents, area, arc length, and surface area.

❖ **Tangent**

Suppose we have a parametric curve: $x = f(t) \quad y = g(t)$. To find the slope of the tangent line at a point on this curve, denoted by $\frac{dy}{dx}$, f and g should be differentiable functions and also y needs to be differentiable with respect to x. Then the Chain Rule gives us: $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$ so

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ if } \frac{dx}{dt} \neq 0$$

<u>**Example 1**</u>: Find the equation of the tangent line to the given curve where $t = \frac{\pi}{4}$.

$$x = \sec t \qquad y = \tan t \qquad -\frac{\pi}{2} < t < \frac{\pi}{2},$$

Need pt: $(x,y) = \left(\sec\frac{\pi}{4}, \tan\frac{\pi}{4}\right) = (\sqrt{2}, 1)$

slope: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

manipulate
12.13

$\frac{dy}{dt} = \sec^2 t \Big|_{t=\frac{\pi}{4}} \quad 2$ $\qquad$ $\frac{dx}{dt} = \sec t \cdot \tan t \Big|_{t=\frac{\pi}{4}} \quad \sqrt{2}$

Tangent line: $y - 1 = \sqrt{2}\,(x - \sqrt{2})$.

Since $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, the curve has a:

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- Horizontal tangent line when $\frac{dy}{dx}=0$ this happens when the numerator is zero:

$$\frac{dy}{dt}=0 \text{ (provided } \frac{dx}{dt}\neq 0\text{)}$$

- Vertical tangent line when $\frac{dy}{dx}$ is undefined this happens when the denominator is zero:

$$\frac{dx}{dt}=0 \text{ (provided } \frac{dy}{dt}\neq 0\text{)}$$

We can use this information to sketch parametric equations. Also finding the second derivative $\left(\frac{d^2y}{dx^2}\right)$ would help us with concavities of the curve. To do that we replace y by $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

WARNING: $\frac{d^2y}{dx^2}\neq\frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$

Example 2: Suppose we have the curve: $x=t-t^2$ $y=t-t^3$

a) For what t values on this curve do we have vertical or horizontal tangent lines?

vertical tangents $\frac{dx}{dt}=0.$

solve: $1-2t=0 \Rightarrow t=\frac{1}{2}$

horizontal tangents $\frac{dy}{dt}=0$

solve: $1-3t^2=0 \Rightarrow t^2=\frac{1}{3} \Rightarrow t=\pm\frac{1}{\sqrt{3}}$

b) Find $\frac{d^2y}{dx^2}$ as a function of t.

① $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1-3t^2}{1-2t}$

② $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{-6t(1-2t)-(-2)(1-3t^2)}{(1-2t)^2}}{1-2t}$

$= \frac{-6t(1-2t)+2(1-3t^2)}{(1-2t)^3}$

❖ **Area**

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geq 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

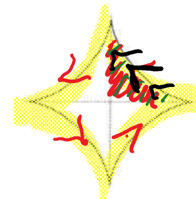
$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt \qquad \left[\text{or } \int_\beta^\alpha g(t)\,f'(t)\,dt\right]$$

$\alpha \to \beta$
or
$\beta \to \alpha$

Note: $\int y\,dx = \int y \frac{dx}{dt}\,dt = \int g(t)\,f'(t)\,dt$

Example 3: Find the area enclosed by the astroid

$x = \cos^3 t \qquad y = \sin^3 t \qquad 0 \le t \le 2\pi$.



$$A = -4\int_0^{\pi/2} \sin^3 t \; 3\cos^2 t \cdot (-\sin t)\, dt$$

$$= +4\int_0^{\pi/2} \sin^4 t \cos^2 t \, dt$$

$$= +4\left(\frac{\pi}{32}\right)$$

$$= +\frac{\pi}{8}$$

(sign due to curves direction)

http://en.wikipedia.org/wiki/Astroid

Cartesian equation:

$x^{2/3} + y^{2/3} = a^{2/3}$

or parametrically:

$x = a\cos^3(t),\ y = a\sin^3(t)$

manipulate 12.16

❖ **Arc Length**

Recall that we defined the arc length of function $y = f(x)$ over an interval as:

The Arc Length formula: If f' is continuous on $[a,b]$, then the length of the curve $y = f(x)$ on $a \le x \le b$ is: $L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$ or as $L = \int_a^b \sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx$ in Leibniz notation.

Now we need to define arc length for a parametric curve:

$$L = \int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$L = \int \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \cdot \frac{dx}{dt}\,dt$$

$$= \int \sqrt{1+\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2} \cdot \sqrt{\left(\frac{dx}{dt}\right)^2}\,dt$$

$$= \int \sqrt{\left(1+\frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}\right)\left(\frac{dx}{dt}\right)^2}\,dt$$

manipulate
12, 15

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Thus if a curve C is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$ where f' and g' are continuous on $[\alpha,\beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is:

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Example 4: Find the length of the circle of radius r defined parametrically by:

$$x = r\cos t;\ y = r\sin t \text{ on } 0 \le t \le 2\pi$$

circumference
$C = 2\pi r$

$$C = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= \int_0^{2\pi} r\sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= \int_0^{2\pi} r \cdot 1\, dt$$

$$= 2\pi r$$

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Manipulate 12.17

Example 5: Find the length of the astroid: $x = \cos^3 t;\ y = \sin^3 t$ on $0 \le t \le 2\pi$

$$L = 4\int_0^{\pi/2} \sqrt{\left[3\cos^2 t(-\sin t)\right]^2 + \left[3\sin^2 t\cos t\right]^2}\,dt$$

$$= 4\int_0^{\pi/2} \sqrt{9\cos^4 t\sin^2 t + 9\sin^4 t\cos^2 t}\,dt$$

$$= 4\int_0^{\pi/2} 3\cos t\sin t\,\underbrace{\sqrt{\cos^2 t + \sin^2 t}}_{1}\,dt$$

$$= 12\int_0^{\pi/2} \cos t\sin t\,dt$$

$$= 12\left[\frac{\sin^2 t}{2}\right]_0^{\pi/2}$$

$$= 12\left(\frac{1}{2}\right)$$

$$= 6.$$

The Astroid: The astroid is not an asteroid. The latter is rock or small planet in outer space. The astroid is a curve that has four points. It derives its name from *astrois,* the Greek word for "star." The curve has had many names, but its modern name was given to it by Joseph Johann von Littrow in 1838. von Littrow was an Austrian astronomer who wrote a widely read book, "Miracles of the Sky." There is a (likely apocryphal) story that von Littrow proposed digging a large circular canal in the Sahara Desert and filling it with burning kerosene. The goal of this was to tell aliens that we are here!

❖ Surface Area

Recall from your book, in the case where f is positive and has a continuous derivative, we define the **surface area** of the surface obtained by rotating the curve $y = f(x), a \le x \le b$, about the x-axis as:

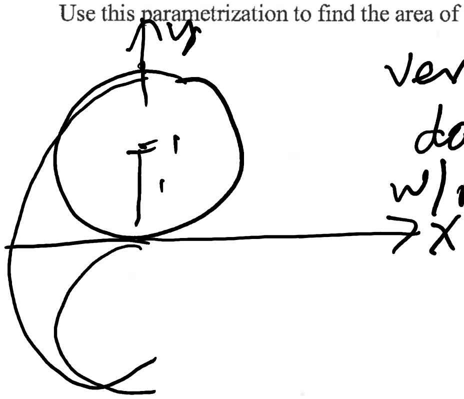
$$S = \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx \text{ or as } S = \int_a^b 2\pi y\sqrt{1+\left[\frac{dy}{dx}\right]^2}\,dx \text{ in Leibniz notation.}$$

In the same way as for arc length, we can obtain a formula for surface area. If the curve given by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$ is rotated about the x-axis, where f', g' are continuous and $g(t) \ge 0$, then the area of the resulting surface is given by:

$$S = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

__Example 6__: The standard parametrization of the circle of radius 1 centered at the point $(0,1)$ in the $xy-$plane is: $x = \cos t;\ y = 1+\sin t$ on $0 \le t \le 2\pi$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis.



vertical donut w/ no hole.

$$S = \int_\alpha^\beta 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dx$$

$y = 1+\sin t$

$\frac{dx}{dt} = -\sin t$

$\frac{dy}{dt} = \cos t$

$$\Rightarrow S = \int_0^{2\pi} 2\pi(1+\sin t)\underbrace{\sqrt{\sin^2 t+\cos^2 t}}_{1}\,dt$$

$$= 2\pi\int_0^{2\pi}(1+\sin t)\,dt$$

$$= 2\pi(2\pi + 0)$$

$$= 4\pi^2 \longleftarrow \text{SA of donut!!}$$