100	90's	80's	70's	60's	<60
1	4	7	7	7	6

Math 163
Fall 2023
Assessment 8
Dusty Wilson

$\bar{x} = 74.0\%$
med $= 75.8\%$

Name: Key

Give me a lever long enough and a fulcrum on which to place it, and I shall move the world.
Archimedes
287-212 BC (mathematician of Syracuse)

No work = no credit

1. Warm-ups

   (a) (1 point) $\vec{k} \times \vec{j} = -\vec{i}$

   (b) (1 point) $\frac{\partial}{\partial y}\sin(x^2 y) = x^2\cos(x^2 y)$

   (c) (1 point) $\int \sin 2x dx = -\frac{1}{2}\cos 2x + C$

2. (1 point) What is the most interesting thing you have done/moved using a lever (see quote above)? Answer using complete English sentences.

   I've been a part of moving decks and sheds using levers and rollers.

3. (8 points) Use a known Maclaurin series to find a power series representation for $\int \cos x^2 dx$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \dots$$

$$\int \cos(x^2)dx = x - \frac{x^5}{5\cdot 2!} + \frac{x^9}{9\cdot 4!} - \frac{x^{13}}{13\cdot 6!} + \dots + C$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{(2n)!(4n+1)}$$

4. (8 points) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n(x-6)^n}{(-7)^n}$

using the ratio test

$$\lim_{n\to\infty}\left|\frac{(n+1)(x-6)^{n+1}}{(-7)^{n+1}} \cdot \frac{(-7)^n}{n(x-6)^n}\right|$$

$$= \lim_{n\to\infty} \frac{n+1}{7n}|x-6|$$

$$= \frac{|x-6|}{7}$$

$\Rightarrow$ solve $\frac{|x-6|}{7} < 1$

$\Rightarrow |x-6| < 7$

ROC: $R = 7$.

5. (8 points) Find a Taylor Series expansion around $x = 1$ for $f(x) = \sqrt[3]{x}. = x^{\frac{1}{3}}$

$f(x) = x^{1/3}\Big|_{x=1} \quad 1$

$f'(x) = \frac{1}{3} x^{-2/3}\Big|_{x=1} \quad \frac{1}{3}$

$f''(x) = \frac{-2}{3^2} x^{-5/3}\Big|_{x=1} \quad -\frac{2}{3^2}$

$f^{(3)}(x) = \frac{10}{3^3} x^{-8/3}\Big|_{x=1} \quad \frac{10}{3^3}$

$\sqrt[3]{x} = 1 + \frac{1}{3}(x-1) - \frac{2}{2!\,3^2}(x-1)^2 + \frac{10}{3!\,3^3}(x-1)^3 - \frac{80}{4!\,3^4}(x-1)^4 + \ldots$

6. (4 points) Find a third-degree Taylor approximation for $f(x) = e^{-3x}$ on the interval $-0.2 \le x \le 0.2$. Then use Taylor's Inequality (aka The Remainder Estimation Theorem) to estimate the accuracy of the approximation $f(x) = T_3(x)$ when x lies on the given interval.

$e^{x} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$

$e^{-3x} = 1 - 3x + \frac{9x^2}{2} - \frac{27}{6}x^3 + \ldots$

$T_3(x) = 1 - 3x + \frac{9x^2}{2} - \frac{9}{2}x^3.$

To estimate the error, we need $f^{(4)}(x) = 81e^{-3x}$

and $|81e^{-3x}| \le \underbrace{81e^{+.6}}_{\text{choose } M = 148}$ on $-0.2 \le x \le 0.2$

so $|R_3(x)| \le \frac{148|x^4|}{4!}$ on $-0.2 \le x \le 0.2$

which has a max ≈ 0.01 on the interval.

Note: I checked graphically, and the exact max error to 3 sig figs is 0.00612