5.3: Diagonalization

Math 220: Linear Algebra

Ex 1: If $D=\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}$ find D^2, D^3, and D^k.

$$D^2 = \begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix} = \begin{bmatrix}3^2 & 0\\ 0 & 4^2\end{bmatrix}$$

$$D^3 = \begin{bmatrix}3^3 & 0\\ 0 & 4^3\end{bmatrix}$$

$$D^k = \begin{bmatrix}3^k & 0\\ 0 & 4^k\end{bmatrix}$$

If $A=PDP^{-1}$ for some invertible *P* and diagonal *D*, then A^k is also easy to compute.

Ex 2: Let $A=\begin{bmatrix}7 & 4\\ -3 & -1\end{bmatrix}$. Find a formula for A^k given that $A=PDP^{-1}$, where

$P=\begin{bmatrix}-2 & -2\\ 3 & 1\end{bmatrix}$ and $D=\begin{bmatrix}1 & 0\\ 0 & 5\end{bmatrix}$

$$P^{-1} = \frac{1}{4}\begin{bmatrix}1 & 2\\ -3 & -2\end{bmatrix} = \begin{bmatrix}\frac{1}{4} & \frac{1}{2}\\ -\frac{3}{4} & -\frac{1}{2}\end{bmatrix}$$

explore: $A^2 = (PD\underbrace{P^{-1})(P}_{I}DP^{-1}) = PD^2P^{-1}$

AND $A^k = PD^kP^{-1}$

$$\Rightarrow A^k = \begin{bmatrix}-2 & -2\\ 3 & 1\end{bmatrix}\begin{bmatrix}1^k & 0\\ 0 & 5^k\end{bmatrix}\begin{bmatrix}\frac{1}{4} & \frac{1}{2}\\ -\frac{3}{4} & -\frac{1}{2}\end{bmatrix}$$

so where did P and D come from?

<u>eigenvalues:</u> solve $0 = \begin{vmatrix}7-\lambda & 4\\ -3 & -1-\lambda\end{vmatrix}$

$$= (7-\lambda)(-1-\lambda)+12$$

$$= \lambda^2 - 6\lambda + 5$$

$$= (\lambda-5)(\lambda-1) \quad \text{so } \lambda = 5 \text{ or } \lambda = 1$$

<u>eigenvectors:</u>

$$\underbrace{\begin{bmatrix}2 & 4\\ -3 & -6\end{bmatrix} \sim \begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}}_{\lambda=5} \Rightarrow \underbrace{\begin{bmatrix}-2\\ 1\end{bmatrix}}_{\text{eigen-vector}} \qquad \underbrace{\begin{bmatrix}6 & 4\\ -3 & -2\end{bmatrix} \sim \begin{bmatrix}1 & \frac{2}{3}\\ 0 & 0\end{bmatrix}}_{\lambda=1} \Rightarrow \underbrace{\begin{bmatrix}\text{XXX <-- -2}\\ \text{XXX <-- 3}\end{bmatrix}}_{\text{eigen-vector}}$$

A square matrix is said to be diagonalizable if A is similar to a diagonal matrix.

Theorem 5 The Diagonalization Theorem
An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

These eigenvectors, since they are linearly independent, form a basis of $\mathbb{R}^n$.

Ex 3: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. That is, find an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$. The eigenvalues are $\lambda = 1, 5$.

we need the eigenvectors.

$\lambda = 1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \;\&\; \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

These are the eigenvectors that go w/ $\lambda = 1$.

$\lambda = 1$ has multiplicity 2.

$\lambda = 5$

$$\begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

This is the eigenvector associated w/ $\lambda = 5$

$\lambda = 5$ has multiplicity 1

$$\Rightarrow A = PDP^{-1} \text{ w/ } P = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Invertible matrix whose columns form an eigenbasis for $\mathbb{R}^n$

Diagonal matrix whose entries are eigenvalues.

Ex 4: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

The eigenvalues are $\lambda = 4$ (algebraic) mult. 2 and $\lambda = 5$

$\underline{\lambda = 4}$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ This is the eigenvector associated w/ $\lambda = 4$.

$\lambda = 4$ has (geometric) multiplicity 1.

$\underline{\lambda = 5}$ This will have one eigenvector.

There aren't enough eigenvectors to diagonalize A

$\therefore$ A is not diagonalizable.

Theorem 6
An $n \times n$ matrix with *n* distinct eigenvalues is diagonalizable.

This is
Not a requirement though for diagonalizable though, as we saw in Ex 3.

Theorem 7
Let *A* be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

see example 4 above

b. The matrix *A* is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals *n*, and this happens if and only if (*i*) the characteristic polynomial factors completely into linear factors and (*ii*) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k.

No complex eigenvalues

c. If *A* is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each *k*, then the total collection of vectors in the sets $B_1, \ldots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Ex 5: Diagonalize the matrix, if possible. $A=\begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Our eigenvalues are $\lambda = 2, 3, 5$

$\lambda = 2$ (Try this 1st as the only way A will not be diagonalizable is if the (geometric) multiplicity is 1.

$$\begin{bmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

multiplicity 2

$\lambda = 3$

$$\begin{bmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 5$

$$\begin{bmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = PDP^{-1} \text{ w/ } P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and thus A is diagonalizable.

Practice Problems

1. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of *A*. Use this information to diagonalize *A*.

3. Let *A* be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if *A* is diagonalizable?

powers of A tell us to diagonalize

5.3: Diagonalization

Practice Problems

1. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

eigenvalues

solve $0 = \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda)+6 = \lambda^2 - 3\lambda + 2$

$\Rightarrow 0 = (\lambda-2)(\lambda-1)$ or $\lambda = 1, 2$

eigenvectors

$\lambda=1$ $\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector associated with $\lambda = 1$.

$\lambda=2$ $\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ eigenvectors associated w/ $\lambda = 2$.

$A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

so $A^8 = PD^8P^{-1} = P\begin{bmatrix} 1^8 & 0 \\ 0 & 2^8 \end{bmatrix}P^{-1}$

$= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix}\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of *A*. Use this information to diagonalize *A*.

we need the eigenvalues.

$\begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ AND $\begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

so the eigenvalues are $\lambda = 1, 3$ and $A = PDP^{-1}$ w/ $P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

3. Let *A* be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if *A* is diagonalizable?

Yes. $\underbrace{\dim(E_{\lambda=3})}_{2} + \underbrace{\dim(E_{\lambda=5})}_{\geq 1} + \underbrace{\dim(E_{\lambda=-2})}_{\geq 1} \geq 4$ but the max it can be is 4

There are enough L.I. eigenvectors to diagonalize.