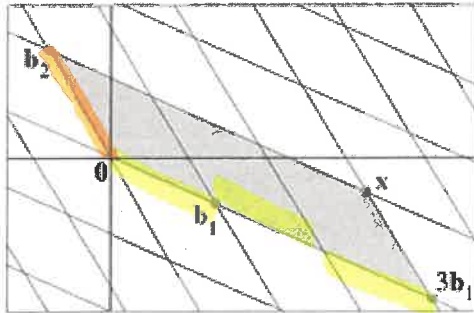


4.7: Change of Basis

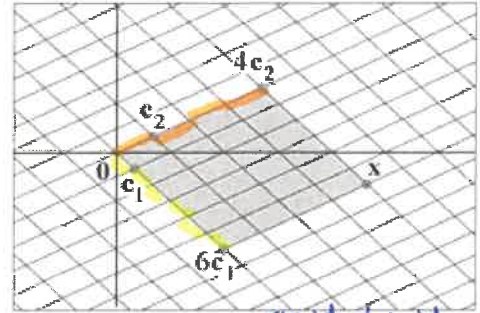
Math 220: Linear Algebra

We are now going to look at converting a vector x in one coordinate system into another coordinate system – same vector, different coordinate representation.

Consider the following vector spaces spanned by $\{\mathbf{b}_1, \mathbf{b}_2\}$ and $\{\mathbf{c}_1, \mathbf{c}_2\}$ respectively.



(a)



(b)

standard grid \vec{x}
parallelogram grid $[\vec{x}]_B$

$$P_B = \begin{bmatrix} 1 & 1 \\ b_1 & b_2 \end{bmatrix}$$

By observation, find

$$[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$3\vec{b}_1 + 1\vec{b}_2$$

and $[\mathbf{x}]_C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$$6\vec{c}_1 + 4\vec{c}_2$$

standard grid \vec{x}
parallelogram grid $[\vec{x}]_C$

$$P_C = \begin{bmatrix} 1 & 1 \\ c_1 & c_2 \end{bmatrix}$$

Ex 1: Consider two bases $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V , such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

Suppose $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$ (that is, $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$), find $[\mathbf{x}]_C$.

$$\Rightarrow [\vec{x}]_C = [3\vec{b}_1 + \vec{b}_2]_C$$

$$= 3[\vec{b}_1]_C + 1[\vec{b}_2]_C$$

$$= \begin{bmatrix} [\vec{b}_1]_C & [\vec{b}_2]_C \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

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Theorem 15

Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $C = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V . Then there is a unique $n \times n$ matrix $P_{C \leftarrow B}$ such that

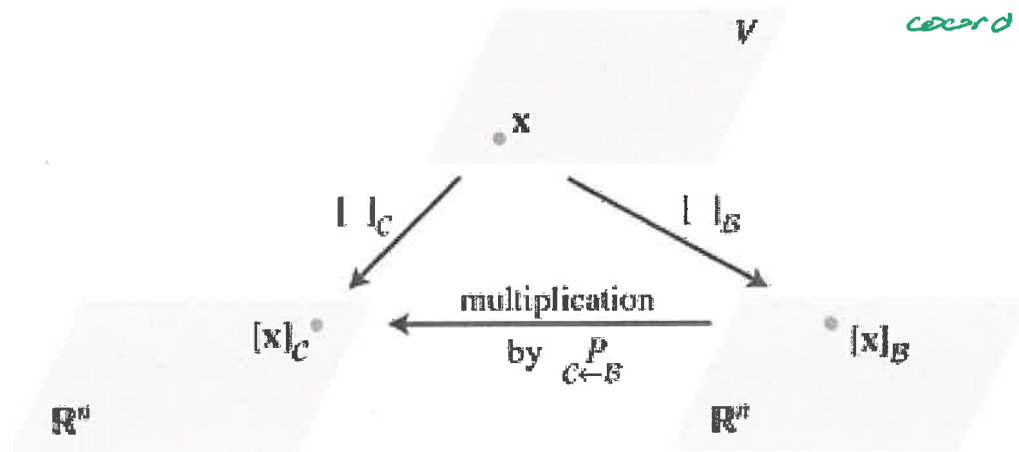
$$[\mathbf{x}]_C = P_{C \leftarrow B} [\mathbf{x}]_B \quad (4)$$

The columns of $P_{C \leftarrow B}$ are the C -coordinate vectors of the vectors in the basis B . That is,

$$P_{C \leftarrow B} = [[\mathbf{b}_1]_C \quad [\mathbf{b}_2]_C \quad \dots \quad [\mathbf{b}_n]_C]$$

Pro tip: To remember this, think of $P_{C \leftarrow B}$ as a linear combination of the columns of $P_{C \leftarrow B}$. The matrix product is a C -coord vector, so the cols should also be C -coord vectors.

$P_{C \leftarrow B}$ is the change of coordinates matrix from B to C of $P_{C \leftarrow B}$



Why are the columns of $P_{C \leftarrow B}$ linearly independent?

The columns form a basis for the vector space \Rightarrow linearly independent.

So $P_{C \leftarrow B}$ is invertible.

So equation (4) above can be re-written as $\begin{pmatrix} P \\ C \leftarrow B \end{pmatrix}^{-1} [\mathbf{x}]_C = [\mathbf{x}]_B$

Since $P_{C \leftarrow B}$ is the matrix that converts B -coordinates to C -coordinates, what should

$\begin{pmatrix} P \\ C \leftarrow B \end{pmatrix}^{-1}$ do?

It should convert from C -coordinates to B -coordinates.

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$$\boxed{\left(\begin{matrix} P \\ C \leftarrow B \end{matrix} \right)^{-1} = \begin{matrix} P \\ B \leftarrow C \end{matrix}}$$

Change of Basis in \mathbb{R}^n

If $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and ε is the *standard basis* $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ in \mathbb{R}^n , then $[\mathbf{b}_1]_{\varepsilon} = \mathbf{b}_1$, and likewise for the other vectors in B . In this case, P is the $\varepsilon \leftarrow B$ same as the change-of-coordinates matrix P_B introduced in Section 4.4, namely,

$$P_B = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \dots \quad \mathbf{b}_n]$$

However, to change coordinates between two non-standard bases in \mathbb{R}^n , we will need to use Theorem 15 and find coordinate vectors of the old basis relative to the new basis.

Ex 2:

Let $\mathbf{b}_1 = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and consider the bases for \mathbb{R}^2 given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from B to C .

we want to find $P_{C \leftarrow B}$

we need \vec{b}_1 & \vec{b}_2 in terms of \vec{c}_1 & \vec{c}_2

$$\Rightarrow \text{solve } \vec{b}_1 = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \vec{b}_2 = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\Rightarrow \text{solve } \begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -6 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 6 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\Rightarrow \text{row reduce } \left[\begin{array}{cc|c} 2 & 6 & -6 \\ -1 & -2 & -1 \end{array} \right] \text{ and } \left[\begin{array}{cc|c} 2 & 6 & 2 \\ -1 & -2 & 0 \end{array} \right]$$

$$\text{which combine to } \left[\begin{array}{cc|cc} 2 & 6 & -6 & 2 \\ -1 & -2 & -1 & 0 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 4 & -2 \\ 0 & 1 & -4 & 1 \end{array} \right]$$

$$\Rightarrow \vec{b}_1 = 9\vec{c}_1 - 4\vec{c}_2 \quad \text{and} \quad \vec{b}_2 = -2\vec{c}_1 + 1\vec{c}_2$$

$$\text{or } [\vec{b}_1]_C = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$$

$$\text{or } [\vec{b}_2]_C = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

conclusion

$$P_{C \leftarrow B} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$$

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The method: In order to find the change of coordinates matrix, row reduce.

$$\left[\begin{array}{cc|cc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{b}_1 & \mathbf{b}_2 \end{array} \right] \sim \left[\begin{array}{cc|cc} I & P \\ & & \mathbf{c}_1 & \mathbf{c}_2 \end{array} \right]$$

New Old

Ex 3: Let $\mathbf{b}_1 = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ and consider the bases for \mathbb{R}^2 given by $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $C = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from C to B.

b. Find the change-of-coordinates matrix from B to C.

$$(a) \left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{array} \right]$$

$$\text{so } P_{B \leftarrow C} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$$

$$(b) \left[\begin{array}{cc|cc} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{array} \right]$$

$$\text{so } P_{C \leftarrow B} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

Practice Problems

1. Let $F = \{\mathbf{f}_1, \mathbf{f}_2\}$ and $G = \{\mathbf{g}_1, \mathbf{g}_2\}$ be bases for a vector space V , and let P be a matrix whose columns are $[\mathbf{f}_1]_G$ and $[\mathbf{f}_2]_G$. Which of the following equations is satisfied by P for all \mathbf{v} in V ?

(i) $[\mathbf{v}]_F = P[\mathbf{v}]_G$

$$P = \begin{bmatrix} | & | \\ [\mathbf{f}_1]_G & [\mathbf{f}_2]_G \\ | & | \end{bmatrix}$$

* (ii) $[\mathbf{v}]_G = P[\mathbf{v}]_F$

This is $P_{G \leftarrow F}$

2. Let B and C be as in Example 1. Use the results of that example to find the change-of-coordinates matrix from C to B .

$$P_{C \leftarrow B} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \Rightarrow P_{B \leftarrow C} = \frac{1}{10} \begin{bmatrix} 1 & 6 \\ -1 & 4 \end{bmatrix}$$

check:

$$\frac{1}{10} \begin{bmatrix} 1 & 6 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

\uparrow \uparrow \uparrow
 $P_{B \leftarrow C}$ $[\mathbf{x}]_C$ $[\mathbf{x}]_B$