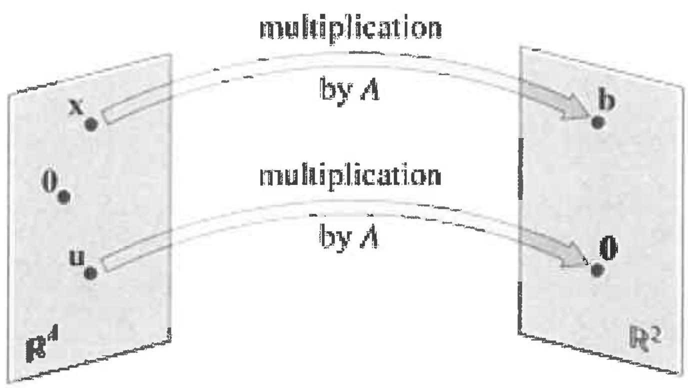
1.8: Linear Transformations

Math 220: Linear Algebra

While the matrix equation $A\vec{x} = \vec{b}$ and the vector equation $x_1\vec{a}_1 + \cdots + x_p\vec{a}_p = \vec{b}$ are essentially the same except for notation, there is a case where the matrix equation represents an action on a vector that isn't directly connected with a linear combination of vectors.

$$\underset{\uparrow \atop A}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\uparrow \atop \mathbf{x}}{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \underset{\uparrow \atop \mathbf{b}}{\begin{bmatrix} 5 \\ 8 \end{bmatrix}} \quad \text{and} \quad \underset{\uparrow \atop A}{\begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}} \underset{\uparrow \atop \mathbf{u}}{\begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}} = \underset{\uparrow}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$



Does this picture look familiar from other math you've seen?

They remind us of a function for matrices and vectors.

1.8: Linear Transformations

A Transformation T from $\mathbb{R}^N$ to $\mathbb{R}^M$ is a rule that assigns each vector $\mathbf{x} \in \mathbb{R}^N$ to a vector $T(\mathbf{x}) \in \mathbb{R}^M$.

The set $\mathbb{R}^N$ is called the Domain of T.

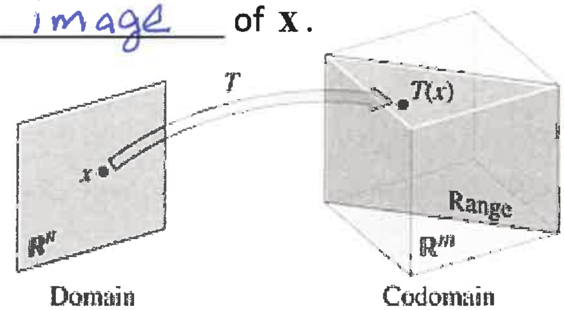
$$T : \mathbb{R}^N \to \mathbb{R}^M$$

The set $\mathbb{R}^M$ is called the codomain of T.

For $\mathbf{x} \in \mathbb{R}^N$, the vector $T(\mathbf{x}) \in \mathbb{R}^M$ is called the image of $\mathbf{x}$.

The set of all images $T(\mathbf{x})$

is called the range of T.



Review Ex. 5 on page 68 of a Rotation Transformation.

An alternate notation.

$$\vec{x} \longmapsto A\vec{x}$$

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Ex 1: Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$,

define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$, so that

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

a. Find $T(\mathbf{u})$, the image of **u** under the transformation *T*.

b. Find an **x** in $\mathbb{R}^2$ whose image under *T* is **b**.

c. Is there more than one **x** whose image under *T* is **b**?

d. Determine if **c** is in the range of the transformation *T*.

a) $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+3 \\ 6-5 \\ -2-7 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix} \leftarrow$ the image of $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ under T.

b) $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{array}{l} x_1 = 3/2 \\ x_2 = -1/2 \end{array} \Rightarrow \vec{x} = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix}$. so $T\begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

c) No. There are no free variables.

d) $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\uparrow$ inconsistent. $0 \neq 1$

$\Rightarrow \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$ is not in the range.

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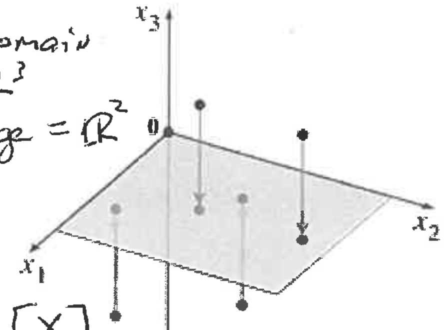
Ex 2: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ *projects* points in $\mathbb{R}^3$ onto the x_1, x_2-plane because

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

codomain $= \mathbb{R}^3$
range $= \mathbb{R}^2$



This can be simplified to $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

codomain $= \mathbb{R}^2$
range $= \mathbb{R}^2$

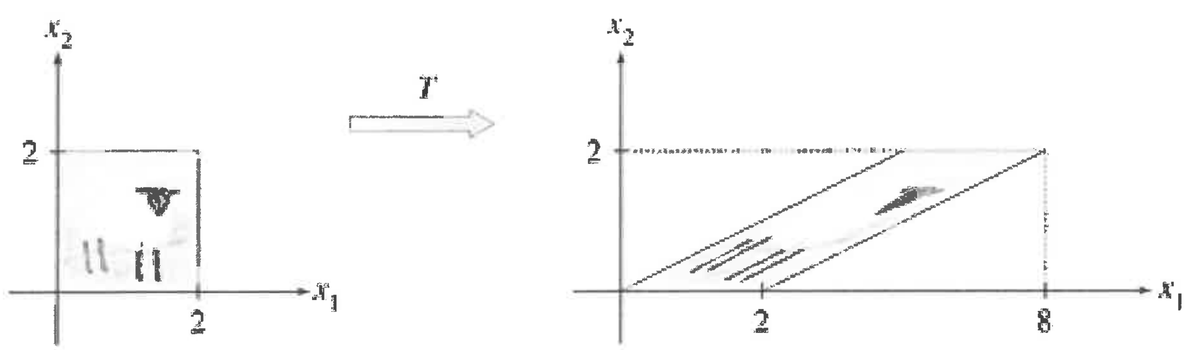
Ex 3: Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. The transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a shear transformation.

For the image below, let's look at the transformations of the vectors $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$T\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

$$T\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$



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Definition
A transformation (or mapping) T is **linear** if:

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all **u**, **v** in the domain of T;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all **u** in the domain of T.

Since the above properties are true for all matrices, then every ___matrix___ transformation is a ___linear___ transformation. (Though the reverse is not true.)

Furthermore, (mini proof)

If T is a linear transformation, then

$$T(\mathbf{0}) = \mathbf{0}$$

$$T(\vec{0}) = T(0\vec{u}) = 0T(\vec{u}) = \vec{0}$$

and

$$T(c\mathbf{u}+d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

$$T(c\vec{u} + d\vec{v}) = T(c\vec{u}) + T(d\vec{v}) = cT(\vec{u}) + dT(\vec{v})$$

for all vectors **u**, **v** in the domain of T and all scalars c, d.

The second property here actually can be generalized to

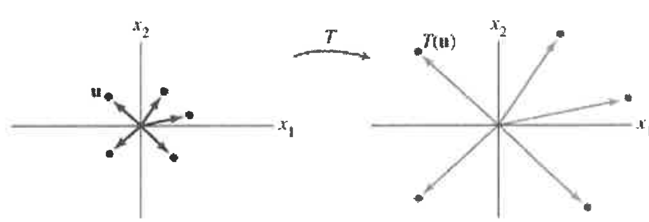
$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p)$$

This is referred to as a ___superposition___ ___principle___ in engineering and physics.

Ex 4: Given a scalar r, define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. T is called a ___contraction___ when $0 \le r \le 1$ and a ___dilation___ when $r > 1$. Let $r = \pi$ and show that T is a linear transformation.

$$T(c\mathbf{u}+d\mathbf{v}) = r(c\vec{u} + d\vec{v})$$
$$= \pi(c\vec{u} + d\vec{v})$$
$$= \pi(c\vec{u}) + \pi(d\vec{v})$$
$$= c(\pi\vec{u}) + d(\pi\vec{v})$$
$$= cT(\vec{u}) + dT(\vec{v})$$

Dilation



$\therefore$ T is a linear Transformation.