4.4-Coordinates-220

4.4: Coordinate Systems

Math 220: Linear Algebra

Theorem 7 The Unique Representation Theorem

claim Let $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \dots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$$

Proof: Let $\vec{x} \in V$ be given. Assume there are two representations for $\vec{x}$ WRT B. So
$\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$ and $\vec{x} = d_1\vec{b}_1 + \dots + d_n\vec{b}_n$
$\Rightarrow \vec{0} = \vec{x} - \vec{x} = (c_1 - d_1)\vec{b}_1 + \dots + (c_n - d_n)\vec{b}_n$
there is only the trivial solution to the homogeneous equation since basis vectors are LI.
$\Rightarrow c_i - d_i = 0$ or $c_i = d_i$ for $i = 1, \dots, n$
$\therefore \vec{x}$ has a unique representation.

Definition
Suppose $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis *B*** (or the ***B*-coordinates of x**) are the weights $c_1, \dots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

We call this vector the coordinate vector of $\vec{x}$ (relative to basis B) or the B-coordinate vector of $\vec{x}$ $\quad [\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

$\mathbf{x} \mapsto [\mathbf{x}]_B$ is the coordinate mapping (determined by B)

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad [\vec{x}]_{polar} = \begin{bmatrix} r \\ \theta \end{bmatrix}$

$\vec{x} \longmapsto [\vec{x}]_{polar}$ mapping
$x = r\cos\theta$
$y = r\sin\theta$

Ex 1: Consider a basis $B = \{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Suppose an $\mathbf{x}$ in $\mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

coord WRT B

$\vec{x} = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ coord WRT $\vec{e}_1, \vec{e}_2$

$[\vec{x}]_B \mapsto \vec{x}$ $\quad \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$

$$\boxed{P_B [\vec{x}]_B = \vec{x}}$$ NEW

$$P_B = \begin{bmatrix} | & | \\ \vec{b}_1 & \vec{b}_2 \\ | & | \end{bmatrix}$$

4.4: Coordinate Systems

Ex 2: The entries in the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ are the coordinates of $\mathbf{x}$ relative to the *standard basis* $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, since

$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\vec{e}_1 + 6\vec{e}_2$

If $\varepsilon = \{\mathbf{e}_1, \mathbf{e}_2\}$, then $[\mathbf{x}]_\varepsilon = \mathbf{x}$.

$[\vec{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

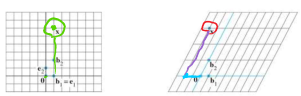


FIGURE 1 Standard graph paper. **FIGURE 2** *B*-graph paper.

$$\begin{tikzcd} \vec{x} \arrow[r, bend right, "P_B^{-1}"'] & {[\vec{x}]_B} \arrow[l, bend right, "P_B"'] \end{tikzcd}$$

See Example 3 on page 219.

Ex 3: Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find the coordinate vector $[\mathbf{x}]_B$ of $\mathbf{x}$ relative to B.

① row reduction

$[\vec{x}]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

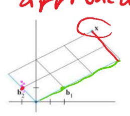
$c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$\Rightarrow \left[\begin{array}{cc|c} 2 & -1 & 4 \\ 1 & 1 & 5 \end{array}\right]$

$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$ so $[\vec{x}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

② Graphical approach



③ change of basis Matrix: P_B^{-1}

$P_B[\vec{x}]_B = \vec{x} \iff [\vec{x}]_B = P_B^{-1}\vec{x}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longleftrightarrow \dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}[\vec{x}]_B = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ $\quad [\vec{x}]_B = \dfrac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

4.4: Coordinate Systems

The matrix P_B changes the B-coordinates of a vector $\mathbf{x}$ into the standard coordinates for $\mathbf{x}$. An analogous change of coordinates can be carried out in $\mathbb{R}^n$ for a basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. Let

$$\boxed{P_B = [\mathbf{b}_1 \ \ \mathbf{b}_2 \ \ \cdots \ \ \mathbf{b}_n]}$$

$$\begin{tikzcd} \vec{x} \arrow[r, bend right, "P_B^{-1}"'] & {[\vec{x}]_B} \arrow[l, bend right, "P_B"'] \end{tikzcd}$$

basis ① L.I. ② span.

Then the vector equation

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n$$

is equivalent to

$$\boxed{\mathbf{x} = P_B[\mathbf{x}]_B} \qquad (4)$$

$[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

P_B = matrix
$[\vec{x}]_B$ = vector

$T(\vec{x}) = A\vec{x}$

$\vec{x} \xrightarrow{A} T(\vec{x}) = A\vec{x}$

matrix mult. (left) gives a linear transformation

We call P_B the **change-of-coordinates matrix** from B to the standard basis in $\mathbb{R}^n$. Left-multiplication by P_B transforms the coordinate vector $[\mathbf{x}]_B$ into $\mathbf{x}$.

Since the columns of P_B form a basis, they are linearly independent, and have an inverse, which leads to

$$P_B^{-1}\mathbf{x} = [\mathbf{x}]_B$$

The Coordinate Mapping

Choosing a basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ for a vector space V introduces a coordinate system in V. The coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ connects the possibly unfamiliar space V to the familiar space $\mathbb{R}^n$. See Figure 5. Points in V can now be identified by their new "names."

$\mathbb{P}_3$ $\quad \mathbb{R}^4$

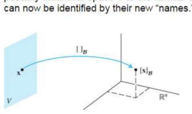


FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

FIGURE 5 The coordinate mapping from V onto $\mathbb{R}^n$.

Theorem 8
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

$\mathbb{P}_3$ $\mathbb{R}^4$

isomorphism between the vector spaces.

$\mathbb{P}_3 \Longleftrightarrow \mathbb{R}^4$

A one-to-one linear transformation from a vector space V onto a vector space W is called an isomorphism from V onto W.

Essentially, these two vector spaces are indistinguishable.

Ex 4: Let B be the standard basis of the space $\mathbb{P}_3$ of polynomials; that is, let $B = \{1, t, t^2, t^3\}$. A typical element $\mathbf{p}$ of $\mathbb{P}_3$ has the form

not a vector $\rightarrow$ $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

Since p is a linear combination of the standard basis vectors, then $[\mathbf{p}]_B = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $\leftarrow$ is a traditional vector.

$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} \Longleftrightarrow 1 + 0t + 2t^2 + 3t^3$

So $\mathbf{p} \mapsto [\mathbf{p}]_B$ is an isomorphism from $\mathbb{P}_3$ onto $\mathbb{R}^4$.

Ex 5: Use coordinate vectors to test the linear independence of the sets of polynomials

a) $1 + 2t^3, 2 + t - 3t^2, -t + 2t^2 - t^3$

$A\vec{x} = \vec{0}$ are there non-triv. sol.

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

$[P_1]_B$ $[P_2]_B$ $[P_3]_B$

$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 2 & 0 \\ 2 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

can't make one poly from the others.

vectors are L.I.

$\Leftrightarrow$ polys are L.I.

$\mathbb{P}_3$

- 1
- t
- t^2
- t^3

$\Big\}$ standard basis

$\dim(\mathbb{P}_3) = 4$

$\mathbb{R}^3$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2, \vec{e}_3$

$\dim(\mathbb{R}^3) = 3$

b) Is this a basis $\mathbb{P}_3$? $(1-t)^2, t - 2t^2 + t^3, (1-t)^3$ $\Longleftrightarrow$ 3 L.I. vectors in $\mathbb{R}^4$

No

3 polys can't span $\mathbb{P}_3$.

Does not span.

$\Rightarrow$ vectors did not span $\mathbb{R}^4$.

$\therefore$ not a basis for $\mathbb{R}^4$

Ex 6: Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$,

and $B = \{\mathbf{v}_1, \mathbf{v}_2\}$. Then B is a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Determine if $\mathbf{x}$ is in H, and if it is, find the coordinate vector of $\mathbf{x}$ relative to B.

① row reducing

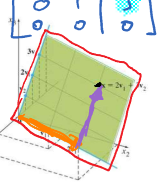
$P_B [\vec{x}]_B = \vec{x}$

$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$ $\leftarrow$ consistent $\vec{x} \in H$

$[\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\vec{x} = 2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$2\vec{v}_1 + 3\vec{v}_2$



$\vec{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ ② Graphing.

$[\vec{x}]_B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\boxed{\text{③ } P_B^{-1} \text{ fails}}$

Practice Problems

1. Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix}$.

a. Show that the set $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis of $\mathbb{R}^3$.

b. Find the change-of-coordinates matrix from B to the standard basis.

c. Write the equation that relates $\mathbf{x}$ in $\mathbb{R}^3$ to $[\mathbf{x}]_B$.

d. Find $[\mathbf{x}]_B$, for the $\mathbf{x}$ given above.

(c.) $P_B [\vec{x}]_B = \vec{x}$

OR

$P_B^{-1} \vec{x} = [\vec{x}]_B$

(a.) $P_B = \begin{array}{c} \begin{array}{ccc} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{array} \\ \begin{bmatrix} 1 & -3 & 3 \\ 0 & 4 & -6 \\ 0 & 0 & 3 \end{bmatrix} \end{array}$ has a pivot in every column. Three L.I. vectors in $\mathbb{R}^3$ must form a basis for $\mathbb{R}^3$.

(d.) $\begin{bmatrix} 1 & 3/4 & 1/2 \\ 0 & 1/4 & 1/2 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} -8 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 1 \end{bmatrix}_B$

$\vec{x}$ $\quad [\vec{x}]_B$

(b.) $\vec{x} \overset{P_B}{\underset{P_B^{-1}}{\rightleftarrows}} [\vec{x}]_B$

so the P_B matrix above does the trick.

$P_B : [\vec{x}]_B \longmapsto \vec{x}$

2. The set $B = \{1 + t, 1 + t^2, t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 6 + 3t - t^2$ relative to B.

$\mathbb{P}_2 \longleftrightarrow \mathbb{R}^3$

so let $P_B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Now, $[\vec{x}]_B = P_B^{-1}\vec{x}$

$= \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}_B$

Thus $p(t) = 5(1+t) + 1(1+t^2) + -2(t+t^2)$.

And $\vec{x} = \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}$

so Let $P_B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Thus $p(t) = 5(1+t) + 1(1+t^2) + -2(t+t^2)$.

AND $\vec{x} = \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}$