Power Series, part 1

❖ **Power Series by the pictures**

Intuitively, a **power series** is like an infinitely long polynomial (except that polynomials are defined so as to have finite length). Examples include:

a.) $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \ldots + (n+1)x^n + \ldots = \sum_{n=0}^{\infty} (n+1)x^n$

$n=0$

b.) $g(x) = x - x^3 + x^5 - x^7 + x^9 - x^{11} + x^{13} - x^{15} + \ldots = \sum_{n=1}^{\infty} (-1)^{n+1} x^{2n-1}$

$g(x) = 0 + x + 0x^2 - x^3 + 0x^4 + x^5 + 0x^6 - x^7 + \ldots$

c.) $h(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 + \frac{1}{6}x^5 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n$

$c_4 \quad c_5 \quad c_n$

Key idea: Working with power series $c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots + c_nx^n + \ldots$ is first and foremost about finding the coefficients $c_0, c_1, c_2, c_3, \ldots, c_n, \ldots$ Often (at least in class) we find a nice formula for the coefficients and simply write $\sum_{n=0}^{\infty} c_n x^n$. But, either way the task at hand begins with finding coefficients.

About now, you might be wondering why anyone would care about power series. The next example provides a graphical connection between power series and more familiar topics.

Example 1: Use a graph to explore $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

1 $\quad \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{1 \cdot 1}{1} + \frac{(-1)x^2}{2}$

$n=0$

2 $\quad \sum \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{1 \cdot x^4}{24}$

$0! = 1$
$1! = 1$
$2! = 2 \cdot 1$
$3! = 3 \cdot 2 \cdot 1$
$4! = 4 \cdot 3 \cdot 2 \cdot 1$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$\vdots$

$$\sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} \quad \vdots$$

$$\sum_{n=0}^{3} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

Polynomial functions can be evaluated using basic operations (addition, subtraction, multiplication and division) and they can be differentiated / integrated pretty easily. But this is not the same for many other functions such as trig, exponential or logarithmic functions! It is beneficial to rewrite a function as a polynomial. This strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions values. Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.

❖ **The Geometric Series**

There are infinitely many power series, but some are famous enough to merit a name. The first of these is named the **geometric series**. The next few examples help us understand this very important (but basic) example.

We will begin with examples without x and then work our way toward actual **power series**

<u>**Example 2**</u>: Evaluate the following:

a.) $\sum_{n=0}^{5} 2\cdot 3^n = 2\cdot 3^0 + 2\cdot 3^1 + 2\cdot 3^2 + 2\cdot 3^3 + 2\cdot 3^4 + 2\cdot 3^5$

$= 728$

b.) $\sum_{n=0}^{5} 2\cdot\left(\frac{1}{3}\right)^n = 2\left(\frac{1}{3}\right)^0 + 2\left(\frac{1}{3}\right)^1 + 2\left(\frac{1}{3}\right)^2 + \ldots + 2\left(\frac{1}{3}\right)^5$

$= 728 \ldots \approx \ldots 999$

b.) $\sum_{n=0} 2\cdot\left(\frac{1}{3}\right)$

$$= \frac{728}{243} \approx 2.999$$

Example 3: The upper limit in the sum is important. Compare $\sum_{n=0}^{50} 2\cdot 3^n$ and $\sum_{n=0}^{50} 2\cdot\left(\frac{1}{3}\right)^n$

$$\sum_{n=0}^{50} 2\cdot 3^n = 2\cdot 3^0 + 2\cdot 3^1 + \dots + 2\cdot 3^{50}$$

big like 3 ⇒ crazy

$\approx 1.44 \times 10^{24}$

$$\sum_{n=0}^{50} 2\left(\frac{1}{3}\right)^n = 2\left(\frac{1}{3}\right)^0 + 2\left(\frac{1}{3}\right) + \dots + 2\left(\frac{1}{3}\right)^{50}$$

small like $\left(\frac{1}{3}\right)$ ⇒ Nice

$\approx 7.79 \times 10^{-24}$

≈ 0

And most interesting is when we allow the upper limit to be infinite in which case we are left with what is called an **infinite series**.

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Example 4: Evaluate the infinite series $\sum_{n=0}^{\infty} 2\cdot\left(\frac{1}{3}\right)^n$

$$\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n = \lim_{k\to\infty} \sum_{n=0}^{k} 2\left(\frac{1}{3}\right)^n$$

$$= \lim_{k\to\infty}\left[2\left(\tfrac{1}{3}\right)^0 + 2\left(\tfrac{1}{3}\right)^1 + 2\left(\tfrac{1}{3}\right)^2 + \dots + 2\left(\tfrac{1}{3}\right)^k\right] 1$$

$$= \lim_{k\to\infty} 2\left[\left(\tfrac{1}{3}\right)^0 + \left(\tfrac{1}{3}\right)^1 + \left(\tfrac{1}{3}\right)^2 + \dots + \left(\tfrac{1}{3}\right)^k\right]\frac{1-\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \lim_{k\to\infty} \frac{2\left[\left(\tfrac{1}{3}\right)^0 + \cancel{\left(\tfrac{1}{3}\right)^1} + \cancel{\left(\tfrac{1}{3}\right)^2} + \dots + \cancel{\left(\tfrac{1}{3}\right)^k} - \cancel{\left(\tfrac{1}{3}\right)^1} - \cancel{\left(\tfrac{1}{3}\right)^2} - \left(\tfrac{1}{3}\right)^3 - \dots - \cancel{\left(\tfrac{1}{3}\right)^{k+1}}\right]}{1-\frac{1}{3}}$$

$$= \lim_{k\to\infty} \frac{2\left[1 - \cancel{\left(\tfrac{1}{3}\right)^{k+1}}\right]}{1-\frac{1}{3}} \quad \to 0$$

$$\sum_{N=0}^{\infty} 2\left(\frac{1}{3}\right)^N = \frac{2(1)}{1-\frac{1}{3}}$$

$$= 2 \div \frac{2}{3} = 2\cdot\frac{3}{2} = 3$$

Reflecting on the previous examples, the following pattern emerges for $\sum_{n=0}^{\infty} a x^n$.

$$\sum_{N=0}^{\infty} a x^N = \frac{a}{1-x} \longleftarrow \text{only valid for small } x\text{'s}$$

The expression $\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}$ is a **power series** (and specifically a **geometric series**). It equals a number when $-1 < x < 1$. It does not equal a number when $|x| \geq 1$. When a series sums to a number, we say, "The series **converges**." When a series does not converge to a number, we say, "The series **diverges**."

Definition: The Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots$$

$$= \frac{1}{1-x}, \quad -1 < x < 1$$

We can modify this formula to find the power series expansion of other functions.

Example 5: Show that $\frac{1}{2-x} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n$ for $|x| < 2$.

$$\frac{1}{2-x} = \frac{1}{2\left(1-\frac{x}{2}\right)} = \frac{\frac{1}{2}}{1-\left(\frac{x}{2}\right)} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^n \cdot x^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n$$

when $\left|\frac{x}{2}\right| < 1$ or $|x| < 2$.

recall $\frac{a}{1-x} = \sum_{n=0}^{\infty} a x^n$

$-1 < x < 1$

$\Leftrightarrow |x| < 1$

Example 6: Find a power series expansion for $f(x) = \frac{1}{1+x^2}$. When does this series converge (equal a number for a given value of x).

$f(x) = \frac{1}{\ldots}$

$\rightarrow f(x) = \sum^{\infty} 1(-x^2)^n$

number for a given value of x).

$f(x) = \frac{1}{1+x^2}$

$= \frac{1}{1-(-x^2)}$

$\to f(x) = \sum_{N=0}^{\infty} 1(-x^2)^N$

$= \sum_{N=0}^{\infty} (-1)^N \cdot x^{2N}$

In the Form: $\sum_{N=0}^{\infty} ax^N = \frac{a}{1-x}$

common ratio.

This converges when $|-x^2| < 1 \Rightarrow |x^2| < 1 \Rightarrow |x| < 1$

Example 7: Integrate the following: $\int \frac{1}{1-8x^3} dx$

$\int \frac{1}{1-8x^3} dx$

$= \int \sum_{N=0}^{\infty} (8x^3)^N dx$

$= \sum_{N=0}^{\infty} \int (8x^3)^N dx$

$8^N x^{3N}$

$= \sum_{N=0}^{\infty} 8^N \int x^{3N} dx$

$= C + \sum_{N=0}^{\infty} \frac{8^N \cdot x^{3N+1}}{3N+1}$

Valid when

$|8x^3| < 1 \Rightarrow |x^3| < \frac{1}{8}$

$\Rightarrow |x| < \sqrt[3]{\frac{1}{8}} = \frac{1}{2}$

So $-\frac{1}{2} < x < \frac{1}{2}$

interval where the power series converges.

We can begin to see the three main aspects of power series come together.

1.) Finding power series representations for functions (and using them to solve questions).
2.) Determining the x values for which the work above is valid. That is, when do the series converge/diverge?
3.) Proving that this whole process is legitimate mathematics.

This last step is (mostly) beyond the scope of Highline mathematics. While we will touch on the middle step, we will leave its finer details for another course. Most of our effort will be spent on methods for finding power series.

As we assumed in the previous example, one of the qualities of power series is that they can be manipulated through addition, subtraction, multiplication, division, differentiation, and integration to find other power series.

For example, assuming the series behave nicely (converge on some neighborhood), we have:

a.) $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n x^n\right]$ (differentiate term by term)

b.) $\int\left[\sum_{n=0}^{\infty} c_n x^n\right] dx = \sum_{n=0}^{\infty}\int c_n x^n\, dx$ (integrate term by term)

Example 8: Show that: $\frac{1}{(1-x)^2} = 1+2x+3x^2+4x^3+5x^4+\ldots$

Notice

① $\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$

$\Rightarrow \int \frac{1}{(1-x)^2}\,dx = \frac{1}{1-x} + C$

observe: $\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$

$= \sum_{n=0}^{\infty} \frac{d}{dx} x^n$

$= \sum_{n=0}^{\infty} n x^{n-1}$

$= 0 + 1\cdot x^0 + 2x^1 + 3x^2 + \ldots$

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$ when $|x| < 1$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots \quad \text{when } |x| < 1$$

Note: Thus far, we are mostly ignoring the question of when these power series are valid.

Example 9: Find a power series representation for $\ln(1-x)$.

$$\frac{d}{dx}\ln(1-x) = \frac{1}{1-x}(-1) = -\frac{1}{1-x}$$

1 or $\int \frac{-1}{1-x}\,dx = \ln(1-x) + C$

$$\Rightarrow \ln(1-x) = C + \int \frac{-1}{1-x}\,dx$$

LHS.

$$= C + \sum_{n=0}^{\infty} -x^n \quad \text{when } -1 < x < 1$$

R.H.S. or $\boxed{|x| < 1}$

ln(2) ln(0)

next: find C

Let $x=0$: L.H.S. $\ln(1) = 0. = $ R.H.S. $C + \sum -0 \Rightarrow C = 0$

Historical note: Today we can evaluate logarithms simply by pushing a calculator button. Prior to that, mathematicians looked up the log values in books. One source of the values in the books was mathematicians evaluating power series like the one above.

Example 10: Integrate $\int \tan^{-1} x\,dx$ using power series

recall:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

From ex. 6.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\int \tan^{-1}x\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty}\int (-1)^n \frac{x^{2n+1}}{2n+1}\,dx$$

$$= C_1 + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)}$$

$\frac{1}{1+x^2} = \sum_{n=0}$

$\Rightarrow \tan^{-1} x = \int \frac{1}{1+x^2}dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$

Find c R.H.S. $= \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx$

set $x=0$

$\tan^{-1}(0) = 0 = c + \Sigma 0$ $= c + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ R.H.S.

$c = 0.$

$= c_1 + \sum_{n=0} (-1) \frac{}{(2n+1)(2n+2)}$

converges when $|x| < 1$

Historical note: Earlier in calculus, we learned to integrate questions like this using integration by parts. Power series provides an alternative approach that requires no calculus skills beyond integrating/differentiating polynomial terms.

Example 11: Solve $f(x) = f'(x)$ using power series.

$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

$\Rightarrow f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots$

$c_0 = c_1$

$c_1 = 2c_2$

$c_2 = 3c_3$

$c_3 = 4c_4$

$\vdots$

$c_n = (n+1) c_{n+1}$

c_0

$c_1 = c_0$

$c_2 = \frac{1}{2} c_1 = \frac{1}{2} c_0$

$c_3 = \frac{1}{3} c_2 = \frac{1}{3} \cdot \frac{1}{2} c_0$

$c_4 = \frac{1}{4} c_3 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} c_0$

$\vdots$

$c_n = \frac{1}{n(n-1)\cdot \ldots \cdot 3 \cdot 2 \cdot 1} c_0 = \frac{1}{n!} c_0$

$f(x) = c_0 e^x$

Thus: $f(x) = c_0\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots\right)$

$$= c_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Note: In this solution we took a term-by-term approach to finding the coefficients $c_0, c_1, c_2, c_3, \dots, c_n, \dots$

This has the same end result as finding a formula $\sum_{n=0}^{\infty} c_n x^n$, it just isn't as concise.

Historical note: You may recognize the question above as a differential equation. Power series provides an extremely powerful technique for solving differential equations that can work on many many questions. It frequently isn't the fastest or cleanest approach … but it works. This is why power series have historically been the "Swiss Army Knife" of functions … they work in a wide variety of situations.

❖ Power Series term by term using derivatives

We have found the derivative of power series. This next example is a little different; here we use derivatives to find the coefficients.

Example 1: Find a power series representation for $f(x) = \sin(x)$.

Suppose $\sin x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$

If True, then true for all x's

If $x=0$: $\boxed{0 = c_0} + c_1(0) + c_2(0) + \ldots$

Differentiate both sides

$\Rightarrow \cos x = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots$

If $x=0$: $\boxed{1 = c_1} + 0 + 0 + \ldots$

Differentiate both sides WRT x

$\Rightarrow -\sin x = 2c_2 + 6c_3 x + 12c_4 x^2 + \ldots$

If $x=0$. $0 = 2c_2 + 0 \Rightarrow \boxed{c_2 = 0}$

Differentiate ...

$\Rightarrow -\cos x = 6c_3 + 24c_4 x + 60c_5 x^2 + \ldots$

If $x=0$: $-1 = 6c_3 \Rightarrow c_3 = \frac{1}{6} = \frac{1}{3 \cdot 2 \cdot 1}$

Ta... derivate

sin 0

cos 0

To date

$c_0 = 0$, $c_1 = \frac{1}{1!}$, $c_2 = 0$; $c_3 = \frac{-1}{3!}$

$\sin 0$
$\cos 0$
$-\sin 0$
$-\cos 0$

$\sin x = \frac{1}{1!}x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

or $\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$

The series we found is an example of a more general type of power series called a **Maclaurin Series**.

Definition: Suppose the function f has derivatives of all orders on an interval centered at $x = 0$, then its **Maclaurin Series** is:

zero

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

This can be written more concisely as: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

Note: A Maclaurin Series is a type of power series. It is found by finding the coefficients term by term using derivatives.

Example 2: Find a Maclaurin Series (that is, a power series) representation for $f(x) = e^x$.

$c_0 = \frac{f(0)}{0!} = \frac{1}{1} = 1$

$c_1 = \frac{f'(0)}{1!} = \frac{1}{1} = 1$

$c_2 = \frac{f''(0)}{2!} = \frac{1}{2!}$

$c_3 = \frac{f^{(3)}(0)}{3!} = \frac{1}{3!}$

$a_4 = \frac{f^{(4)}(0)}{4!} = \frac{1}{4!}$

$c_n = \frac{1}{n!}$

Thus

$x \quad \dots \quad x^3 \quad x^4 \quad x^5$

Thus

$e^x = 1 + 1x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{x^4}{4!} + \frac{x^5}{5!} \dots$

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$\leftarrow$ included in the soln to $f(x) = f'(x)$

As with geometric series, the Maclaurin Series can be manipulated to go quite a way:

Example 3: Find a Maclaurin Series (that is, a power series) representation for the following:

a.) $f(x) = e^{2x}$

recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$

b.) $g(x) = x^4 e^x$

recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow x^4 e^x = x^4 \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$= \sum x^4 \frac{x^n}{n!} = \sum \frac{x^{n+4}}{n!}$

$$= \sum_{n=0}^{\infty} \frac{x^4 x^n}{n!} = \sum_{n=0} \frac{x}{n!}$$

Three Maclaurin Series to Memorize:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

odd fct   even fct

If you have been paying attention, you may have noticed that we have found power series in the previous few examples, but have NOT made mention of where these series are valid. This is because we are no longer working with geometric series and consequently need more power (pun).

Specifically, we will use two facts and one method/test.

Fact 1: The Harmonic Series diverges. In symbols: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \infty$$

not important

Fact 2: The *Alternating* Harmonic Series converges. In symbols: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \text{ converges}$$

So why does one infinite series converge and another diverge? That is a big question and we will leave it for another course. However, we can see that sometimes we sum an infinite number of terms and get a number (converge) and other times we don't get a number (diverge).

One of the most powerful ways of determining if a series will converge is to ask, "Do the terms decrease fast enough to converge? But how do we measure, "Fast enough"?

Example 4: Explore the ratio of consecutive terms on these three series

a.) $\sum_{n=0}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^n = 2 + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^4 + \cdots$

ratio: $\frac{2\left(\frac{1}{3}\right)^4}{} = \frac{1}{}$

geometric series

$$\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n = \frac{2}{1-1/3} = 3$$

ratio: $\frac{2\left(\frac{1}{3}\right)^4}{2\left(\frac{1}{3}\right)^3} = \frac{1}{3}$

notice: $\frac{a_{n+1}}{a_n} = \frac{1}{3}$

b.) $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{n} + \frac{1}{n+1} + \dots$

$$\frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} \qquad \frac{\frac{1}{6}}{\frac{1}{5}} = \frac{5}{6} \qquad \frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$$

As $n \to \infty$ $\frac{n}{n+1} = 1$

c.) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \cancel{x} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \dots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \dots$

$$\frac{\frac{x^3}{3!}}{\frac{x^2}{2!}} = \frac{x}{3} \qquad \frac{\frac{x^6}{6!}}{\frac{x^5}{5!}} = \frac{x}{6} \qquad \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1}$$

As $n \to \infty$ for a fixed value of x, we know $\frac{x}{n+1} \longrightarrow 0$

As with geometric series, the Maclaurin Series can be manipulated to go quite a way:

<u>Example 3</u>: Find a Maclaurin Series (that is, a power series) representation for the following:

a.) $f(x) = e^{2x}$

b.) $g(x) = x^4 e^x$

<u>Three Maclaurin Series to Memorize</u>:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

❖ **A deep dive into where Power Series Converge**

If you have been paying attention, you may have noticed that we have found power series in the previous few examples, but have NOT made mention of where these series are valid. This is because we are no longer working with geometric series and consequently need more power (pun).

Specifically, we will use two facts and one method/test.

<u>Fact 1</u>: The Harmonic Series diverges. In symbols: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \infty$$

Harmonic series diverges.

<u>Fact 2</u>: The *Alternating* Harmonic Series converges. In symbols: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(2)$

$$\sum^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

So why does one infinite series converge and another diverge? That is a big question and we will leave it for another course. However, we can see that sometimes we sum an infinite number of terms and get a number (converge) and other times we don't get a number (diverge).

One of the most powerful ways of determining if a series will converge is to ask, "Do the terms decrease fast enough to converge? But how do we measure, "Fast enough"?

Example 4: Explore the ratio of consecutive terms on these series

a.) $\sum_{n=0}^{\infty} 2\cdot\left(\frac{1}{3}\right)^n = 2\cdot\left(\frac{1}{3}\right)^0 + 2\cdot\left(\frac{1}{3}\right)^1 + 2\cdot\left(\frac{1}{3}\right)^2 + 2\cdot\left(\frac{1}{3}\right)^3 + 2\cdot\left(\frac{1}{3}\right)^4 + \ldots + 2\cdot\left(\frac{1}{3}\right)^n + 2\cdot\left(\frac{1}{3}\right)^{n+1} + \ldots$

Converges

$\frac{2(\frac{1}{3})^1}{2(\frac{1}{3})^0} = \frac{1}{3}$ $\qquad$ $\frac{2(\frac{1}{3})^4}{2(\frac{1}{3})^3} = \frac{1}{3}$ $\qquad$ $\frac{2(\frac{1}{3})^{n+1}}{2(\frac{1}{3})^n} = \frac{1}{3}$

and it appears that as $n \to \infty$, the ratio of consecutive terms Approaches $\frac{1}{3}$

b.) $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} + \frac{1}{n+1} + \ldots$

Diverges

$\frac{\frac{1}{2}}{\frac{1}{1}} = \frac{1}{2}$ $\qquad$ $\frac{\frac{1}{5}}{\frac{1}{4}} = \frac{4}{5}$ $\qquad$ $\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1} \longrightarrow 1$

and it appears that as $n \to \infty$, the ratio of consecutive terms approaches 1

c.) $\sum_{n=0}^{\infty} 2\cdot 3^n = 2\cdot 3^0 + 2\cdot 3^1 + 2\cdot 3^2 + 2\cdot 3^3 + 2\cdot 3^4 + \ldots + 2\cdot 3^n + 2\cdot 3^{n+1} + \ldots$

Diverges

$\frac{2\cdot 3^1}{2\cdot 3^0} = 3$ $\qquad$ $\frac{2\cdot 3^4}{2\cdot 3^3} = 3$ $\qquad$ $\frac{2\cdot 3^{n+1}}{2\cdot 3^n} = 3$

and it appears that as $n \to \infty$, the ratio of consecutive terms approaches 3

d.) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots + \frac{(-1)^{n+1}}{n} + \frac{(-1)^{n+2}}{n+1} + \ldots$

d.) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots + \frac{(-1)^{n+1}}{n} + \frac{(-1)^{n+2}}{n+1} + \ldots$

Converges

$\left|\frac{-\frac{1}{2}}{\frac{1}{1}}\right| = \left|-\frac{1}{2}\right|$ $\quad \left|\frac{\frac{1}{5}}{-\frac{1}{4}}\right| = \left|-\frac{4}{5}\right|$ $\quad \left|-\frac{\frac{1}{N+1}}{\frac{1}{N}}\right| = \left|-\frac{N}{N+1}\right| \to 1$

and it appears that as $n \to \infty$, the ratio of consecutive terms approaches 1

e.) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots$

converges to e^x.

$\left|\frac{x}{1}\right|$ $\quad \left|\frac{\frac{x^4}{4!}}{\frac{x^3}{3!}}\right| = \left|\frac{3!\,x}{4!}\right| = \left|\frac{x}{4}\right|$ $\quad \left|\frac{\frac{x^{N+1}}{(N+1)!}}{\frac{x^N}{N!}}\right| = \left|\frac{x}{N+1}\right| = \frac{1}{N+1}|x| \to 0$

and it appears that as $n \to \infty$, the ratio of consecutive terms approaches 0

?

Let's sum up what we have seen about series and ratios thus far.

1. Power series, coefficients, and terms
   a. A power series is of the form: $\sum_{n=0}^{\infty} c_n x^n = \underbrace{c_0}_{a_0} + \underbrace{c_1 x}_{a_1} + \underbrace{c_2 x^2}_{a_2} + \underbrace{c_3 x^3}_{a_3} + \ldots + \underbrace{c_n x^n}_{a_n} + \underbrace{c_{n+1} x^{n+1}}_{a_{n+1}} + \ldots$

      NO x's

      include x's

   b. The coefficients are: $c_0, c_1, c_2, c_3, \ldots, c_n, c_{n+1}, \ldots$
   c. The terms are: $a_0, a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$
2. About ratios of consecutive terms a_n and a_{n+1}:
   a. Ratios > 1 mean the terms are increasing quickly.
   b. Ratios < 1 mean the terms are decreasing to zero rapidly.
   c. Ratios $\approx$ 1 aren't changing quickly enough to know to draw a conclusion (using ratios).
3. About absolute value
   a. Terms can vary in sign. The absolute value of the ratio sometimes makes quantities bigger (positive) and thus convergence more difficult. So if a series converges with the absolute value, it certainly converges without it.

This leads us to a powerful and (relatively) easy test for convergence.

<u>Definition</u>: The Ratio Test

a.) If $\lim \left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum_{}^{\infty} a_n = \sum_{}^{\infty} c_n x^n$ is (absolutely) convergent

Definition: The Ratio Test

a.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n x^n$ is (absolutely) convergent.

b.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n x^n$ divergent.

c.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive. Try again!

Review the questions on the previous page. Notice how they provide examples of the Ratio Test at work. In particular, notice that when the ratio approaches 1 this can mean either convergence or divergence. That is why we call the test "inconclusive" in this instance.

Example 4: Where does the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Ratio Test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|$

$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right|$

$= 0 \leftarrow$ for all x.

The series converges for values of x. $(-\infty, \infty)$.

Example 5: Consider $\frac{5}{4-x} = \frac{5}{1-(-3-x)}$ with geometric series $\sum_{n=0}^{\infty} 5(-3-x)^n = \sum_{n=0}^{\infty} 5(-1)^n (3+x)^n$.

Find the x values where this series converges using the ratio test.

ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{5}(-1)^{n+1}(3+x)^{n+1}}{\cancel{5}(-1)^n(3+x)^n}\right|$

$= \lim_{n\to\infty} |3+x|$

Series converges when $|3+x| < 1$

$\boxed{\text{Test } x = -2 \text{ Diverge}}$

Test

Series converges when $|3+x| < 1$

$\Rightarrow -1 < 3+x < 1$

$\Rightarrow -4 < x < -2$

Test
$x = -4$
Diverge

As we focus in on convergence, two definitions will help us.

Definition: The **interval of convergence** of a power series is the interval that consists of all values of x for which a series converges.

Note: Intervals have **endpoints** that may (or may not) be included in the interval of convergence.

Lazy Definition: The **radius of convergence** is half the width of the interval of convergence (possibly zero or infinity).

Example 5 revisited: Consider $\frac{5}{4-x} = \sum_{n=0}^{\infty} 5(-1)^n (3+x)^n$ and find its interval of convergence and radius of convergence.

width = 2

I.O.C. $\quad -4 < x < -2 \quad$ or $\quad (-4, -2)$

R.O.C. $\quad R = \frac{2}{2} = 1$

The following examples show how we find the interval of convergence and radius of convergence once we have a power series in hand.

Example 6: Suppose you are given a power series $\sum_{n=1}^{\infty} \frac{x^n}{n\,3^n}$. Find its interval of convergence and radius of

Example 6: Suppose you are given a power series $\sum_{n=1}^{\infty} \frac{x^n}{n\,3^n}$. Find its interval of convergence and radius of convergence.

ratio test: $\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)3^{n+1}}}{\frac{x^n}{n3^n}} \right| = \lim_{n\to\infty} \left| \frac{x \cdot n}{3(n+1)} \right| = \left| \frac{x}{3} \right|$

conclusion
I.o.c: $[-3, 3)$
R.o.c: $R = 3$

Solve $\left|\frac{x}{3}\right| < 1$

$\Rightarrow |x| < 3$

$\Rightarrow -3 < x < 3$

$\boxed{\text{Test } x = -3}$

$\sum_{n=1}^{\infty} \frac{(-3)^n}{n\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges

$\boxed{\text{Test } x = 3}$

$\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Example 7: Suppose you are given a power series $\sum_{n=0}^{\infty} \frac{(-3)^n x^{2n}}{\sqrt{n+1}}$. Find its ~~interval of convergence~~ and radius of convergence.

ratio test: $\lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1} x^{2(n+1)}}{\sqrt{n+2}}}{\frac{(-3)^n x^{2n}}{\sqrt{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(-3)^{n+1} x^{2n+2}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^{2n}} \right|$

$= \lim_{n\to\infty} \left| \frac{3x^2 \sqrt{n+1}}{\sqrt{n+2}} \right|$

$= |3x^2| < 1$

$\Rightarrow |x^2| < \frac{1}{3}$

$\Rightarrow |x| < \frac{1}{\sqrt{3}}$

$\Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

, end points are beyond us

• R.O.C. $R = \frac{1}{\sqrt{3}}$

Example 8: Find the interval of convergence and radius of convergence of the Maclaurin series

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

ratio test: $\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$

$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+3)(2n+1)} \right|$

$= 0 < 1$

I.O.C. $(-\infty, \infty)$

R.O.C. $R = \infty$.

Example (7.1): Find the Taylor Series for $f(x) = \frac{1}{\sqrt{x}}$ centered at $\cancel{a=9}$ $a = 16$.

	At $x = 16$
$f(x) = x^{-1/2}$	$+\frac{1}{4} = \frac{1}{2^2}$
$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$	$-\frac{1}{2}\cdot\frac{1}{4^3} = -\frac{1}{2^7}$
$f''(x) = +\frac{1\cdot 3}{2^2}x^{-5/2}$	$+\frac{1\cdot 3}{2^2}\cdot\frac{1}{4^5} = +\frac{1\cdot 3}{2^{12}}$
$f'''(x) = -\frac{1\cdot 3\cdot 5}{2^3}x^{-7/2}$	$-\frac{1\cdot 3\cdot 5}{2^3}\cdot\frac{1}{4^7} = -\frac{1\cdot 3\cdot 5}{2^{17}}$
$f^{(4)}(x) = +\frac{1\cdot 3\cdot 5\cdot 7}{2^4}x^{-9/2}$	$+\frac{1\cdot 3\cdot 5\cdot 7}{2^4}\cdot\frac{1}{4^9} = +\frac{1\cdot 3\cdot 5\cdot 7}{2^{22}}$

$$\Rightarrow \frac{1}{\sqrt{x}} = \frac{1}{2^2} - \frac{\frac{1}{2^7}}{1!}(x-16) + \frac{\frac{1\cdot 3}{2^{12}}}{2!}(x-16)^2 - \frac{\frac{1\cdot 3\cdot 5}{2^{17}}}{3!}(x-16)^3 + \frac{\frac{1\cdot 3\cdot 5\cdot 7}{2^{22}}}{4!}(x-16)^4 + \cdots$$

continue on in the same manner.

❖ Taylor Series and Approximations

We are learning about power series. We have done this in two steps:

- We learned to find power series.
- We found where those series were convergent.

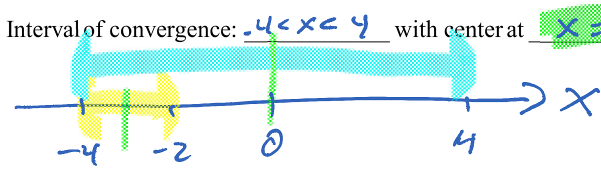
This second point and the interval of convergence provides us a clue toward our next step.

Example (8.1): Review two power series for $\frac{5}{4+x}$

$\frac{5}{4+x}=\sum_{n=0}^{\infty}5(-1)^n(3+x)^n$ ← Interval of convergence: $-4<x<-2$ with center at $x=-3$

and also

$\frac{5}{4+x}=\sum_{n=0}^{\infty}\frac{5}{4}\left(-\frac{x}{4}\right)^n$ ← Interval of convergence: $-4<x<4$ with center at $x=0$



Notice that one function may have more than one power series representation. These series have different intervals of convergence. Of particular note, the center of the intervals of convergence are different.

Historical Note: One of the most innovative aspects of Cauchy's [limit focused] program of rigor was his rejection of divergent series. These had been widely used in the eighteenth century, before Cauchy declared that they were unacceptably ill-defined, and produced ambiguous or even erroneous results. Picking up on this point in several papers of the early 1830s, Poisson tried to come to a clearer understanding of these series and the boundaries of their legitimacy. In his 1844 paper ''On Divergent Series and Various Points of Analysis Connected with Them,'' De Morgan blasted not only Poisson's ideas and Cauchy's definition of the integral on which they were based, but the whole preoccupation with certainty which valorized the search for rigor. ''Divergent series, at the time Poisson wrote, had been nearly universally adopted for more than a century, and it was only here and there that a difficulty occurred in using them,'' he fumed. The knowledgeable mathematician, De Morgan pointed out, could easily detect and correct such problems when they arose. To artificially control their use just in order to guarantee rigorous exactitude was at best unnecessary and ridiculous. At worst it could stand in the way of deeper understanding of the truth embodied in these series, which was as yet still poorly comprehended. As De Morgan wrote: "We must admit that many series are such as we cannot at present safely use, except as means of discovery, the results of which are to be subsequently verified. But to say that what we cannot use no others ever can, to refuse that faith in the future prospects of algebra which has already realized so brilliant a harvest . . . seems to me a departure from all rules of prudence." For De Morgan, to draw back from poorly defined or understood mathematical conclusions was a grievous error.[1]

[1] Joan Richards (2011) God, Truth and Mathematics in Nineteenth-century England, Theology and Science, 9:1, 53-74, DOI: 10.1080/14746700.2011.547005

This is not the first time we have generated a power series with a known center point. In particular, the Maclaurin Series formula $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ generates series centered at $x=0$. This formula (and the accompanying derivation) can be modified to generate series centered at $x=a$.

Maclaurin Series formula $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ generates series centered at $x=0$. This formula (and the accompanying derivation) can be modified to generate series centered at $x=a$.

<u>Definition</u>: If *f* has a power series representation (expansion) at $x=a$, that is if $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ when $|x-a|<R$, then its coefficients are given by the formula $c_n=\frac{f^{(n)}(a)}{n!}$. We call the series above a Taylor Series.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \leftarrow \text{Maclaurin Series}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \leftarrow \text{Taylor Series}$$

<u>Example (8.2)</u>: Find the Taylor Series for $f(x)=\frac{1}{\sqrt{x}}$ centered at $a=16$

$$\frac{1}{\sqrt{x}}=\frac{1}{2^2}-\frac{\frac{1}{2^7}}{1!}(x-16)+\frac{\frac{1\cdot 3}{2^{12}}}{2!}(x-16)^2-\frac{\frac{1\cdot 3\cdot 5}{2^{17}}}{3!}(x-16)^3+\frac{\frac{1\cdot 3\cdot 5\cdot 7}{2^{22}}}{4!}(x-16)^4+\cdots$$

continue on in the same manner.

Example 3: Find the Taylor Series for $g(x) = x - x^3$ centered at $a = 2$

	@ $a=2$
$g(x) = x - x^3$	-6
$g'(x) = 1 - 3x^2$	-11
$g''(x) = -6x$	-12
$g^{(3)}(x) = -6$	-6
$g^{(4)}(x) = 0$	0
$\vdots$	$\vdots$

$$g(x) = x - x^3$$

$$= \frac{-6}{0!} - \frac{11(x-2)}{1!} - \frac{12(x-2)^2}{2!} - \frac{6}{3!}(x-2)^3 + 0 \ldots$$

Example 2 revisited: Answer the following for $f(x) = \dfrac{1}{\sqrt{x}}$

a.) Find the Taylor polynomials T_0, T_1, T_2, T_3 centered at $x = 16$.

$T_0 = \dfrac{1}{2^2}$

T_N

$T_1 = \dfrac{1}{2^2} - \dfrac{\frac{1}{2^7}}{1!}(x-16)$

$T_2 = \dfrac{1}{2^2} - \dfrac{\frac{1}{2^7}}{1!}(x-16) + \dfrac{\frac{1\cdot 3}{2^{12}}}{2!}(x-16)^2$

Interesting math side note on how to write the product of evens or odds. Here are a couple of little examples:

Evens: $\overset{2\cdot 6}{12} \times \underset{2\cdot 5}{10} \times \overset{2\cdot 4}{8} \times \underset{2\cdot 3}{6} \times \overset{2\cdot 2}{4} \times \underset{2\cdot 1}{2} = 2^6 6!$

the odds are a little more difficult

Odds:

$1\cdot 3\cdot 5\cdot 7\cdot 9\cdot \ldots$

$T_2 = \frac{1}{2^2} - \frac{\frac{1}{2^7}}{1!}(x-16) + \frac{\frac{1}{2^{12}}}{2!}(x-16)^2$

$T_3 = \frac{1}{2^2} - \frac{1}{2^7 \cdot 1!}(x-16) + \frac{1\cdot 3}{2^{12}\cdot 2!}(x-16)^2 - \frac{1\cdot 3\cdot 5}{2^{17}\cdot 3!}(x-16)^3$

the odds are a little more difficult

Odds:

$$9\times7\times5\times3\times1 = \frac{9\times8\times7\times6\times5\times4\times3\times2\times1}{8\times6\times4\times2} = \frac{9!}{2^4 4!}$$

b.) Use T_0, T_1, T_2, T_3 to approximate $\frac{1}{\sqrt{16.8}}$ and $\frac{1}{\sqrt{22}}$. Then compare these approximations to the calculator value and list the errors. Note that absolute error is the difference between function value and the value that the Taylor Polynomial provides us.

			error	
T_0	0.25	0.25	0.006	0.0368
T_1	0.2438	0.2031	0.0002	0.0101
T_2	0.2440	0.2163	0.00002	0.0031
T_3	0.2440	0.2122	0.00002	0.0010
	$\frac{1}{\sqrt{16.8}}$ 0.2440	$\frac{1}{\sqrt{22}}$ 0.2132	$\frac{1}{\sqrt{16.8}}$ error	$\frac{1}{\sqrt{22}}$ error

error = |Actual − Approx|

small error

larger error

Taylor Series centered @ $x = 16$

$\frac{1}{\sqrt{16.8}} \to x = 16.8$

$\frac{1}{\sqrt{22}} \to x = 22$

There is a problem. Taylor Polynomials (and Power Series) allow us to generate approximate values. However, these are *estimates*. How close are the approximate values to the actual values? In order to find the error, we would need to know the exact value but if we knew the exact values we would not need an estimate in the first place.

the error, we would need to know the exact value but if we knew the exact values we would not need an estimate in the first place.

We need a way to find the error that does not require that we know the exact value. Or, to be more precise, we need a way to bound the error.

This requires that we build up more notation.

Definition: Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I, $f(x) = T_n(x) + R_n(x)$ where T_n is the nth-degree Taylor Polynomial for f centered at $x = a$ and $R_n(x)$ is the remainder.

To be clear: $\underbrace{f(x)}_{\substack{\text{exact}\\ \text{value}}} = \underbrace{T_n(x)}_{\substack{\text{approximate}\\ \text{value}}} + \underbrace{R_n(x)}_{\substack{\text{remainder}\\ \text{or error}}}$

Example 2 revisited: Find $R_3(x)$ for the 3rd degree Taylor Polynomial of $f(x) = \dfrac{1}{\sqrt{x}}$ centered at $x = 16$.

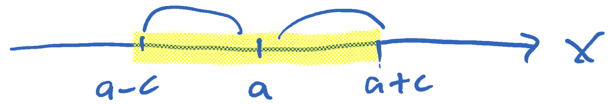
$$\frac{1}{\sqrt{x}} = \underbrace{\frac{1}{2^2} - \frac{1}{2^7 \cdot 1!}(x-16) + \frac{1\cdot 3}{2^{12}\cdot 2!}(x-16)^2 - \frac{1\cdot 3\cdot 5}{2^{17}\cdot 3!}(x-16)^3}_{T_3} + \underbrace{\frac{1\cdot 3\cdot 5\cdot 7}{2^{22}\cdot 4!}(x-16)^4 - \dots}_{R_3 \to}$$

$$R_3(x) = \frac{1\cdot 3\cdot 5\cdot 7}{2^{22}\cdot 4!}(x-16)^4 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^{27}\cdot 5!}(x-16)^5 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{2^{32}\cdot 6!}(x-16)^6 - \dots$$

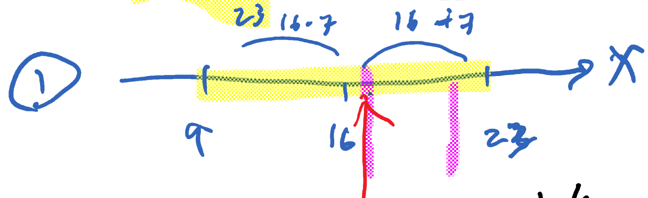
remainder or error

Notice: Finding $R_3(x)$ requires knowing the exact value of $\dfrac{1}{\sqrt{x}}$. In this case we can use our calculator to evaluate $\dfrac{1}{\sqrt{x}}$, but what if we didn't have that ability?

The Remainder Estimation Theorem: Suppose there exists a number M such that $\left|f^{(n+1)}(x)\right| \le M$ for all x in the interval $[a-c, a+c]$. The remainder of the nth-degree Taylor Polynomial for f centered at a satisfies: $|R_n(x)| \le M \dfrac{|x-a|^{n+1}}{(n+1)!}$



Example 2 revisited: Bound the error in T_3 on the interval $9 \le x \le \cancel{22}$ 23 and find the upper bound (max) error in your estimates for $\dfrac{1}{\sqrt{16.8}}$ and $\dfrac{1}{\sqrt{22}}$.

① 

② Find M.

$$\left|\underbrace{f^{(3+1)}(x)}\right| \le M \text{ on } [9, 23]$$

$$\Rightarrow \left|\frac{-3\cdot5\cdot7\,x^{-9/2}}{16}\right| \le M$$

$$\Rightarrow \left|\frac{105}{16\,x^{9/2}}\right| \le M \text{ on } [9, 23]$$

greatest when x is smallest ($x=9$)

Scratch

$f \quad x^{-1/2}$

$f' \quad -\frac{1}{2}x^{-3/2}$

$f'' \quad \frac{3}{4}x^{-5/2}$

$f''' \quad \frac{3\cdot5}{8}x^{-7/2}$

$f^{(4)}(x) = -\frac{3\cdot5\cdot7}{16}x^{-9/2}$

Let $M = \dfrac{105}{16(9)^{9/2}} \approx 0.0003334$

③ Bound the remainder/error.

$$|R_3(x)| \le M\frac{|x-16|^4}{4!}$$

evaluate when $x = 16.8$ and $x = 22$

① $|R_3(x)| \le \dfrac{0.0003334\,|16.8-16|^4}{4!}$

≈ 0.00000569 (very small error).

② $|R_3(x)| \le 0.0003334\,|22-16|^4$

② $|R_3(x)| \le \frac{0.0003334\,|22-16|^4}{4!}$

$= 0.018$ (larger max error).

Example 3: Approximate $\int_0^1 e^{-x^2}\,dx$ to within 0.001

① recall: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$

② $\int e^{-x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}dx$

$= \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{n!}dx$

$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$

③ $\int_0^1 e^{-x^2}dx = \left[C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\right]_0^1$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$ ← exact

④ Approx w/in 0.001. Alternating series estimation theorem.

$\int_0^1 e^{-x^2}dx = +1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} + \cdots$

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!} - \frac{1}{11\cdot 5!} + \cdots$$

$$\underbrace{1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}}_{0.7475}$$

0.7475 ± 0.0008

Bonus Questions

Example 4: Find the first three non-zero terms in the Maclaurin series for $f(x) = e^x \sin x$.

① recall: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

② multiply. $e^x \sin x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots$

$- \frac{x^3}{3!} - \frac{x^4}{3!} - \frac{x^5}{3!} + \cdots$

$+ \frac{x^5}{5!} + \frac{x^6}{5!} + \frac{x^7}{5!} + \cdots$

③ Add $e^x \sin x \approx x + x^2 + \frac{1}{3}x^3 + 0x^4 - \frac{19}{120}x^5 + \cdots$

Example 5: Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

Ratio test: $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)^{n+1}}}{\frac{(x-2)^n}{n^n}} \right|$

$= \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{(x-2)^n} \right|$

$= \lim_{n\to\infty} \left| \frac{(x-2) \cdot n^n}{(n+1)^{n+1}} \right|$

$$= \lim_{n\to\infty} \left| \frac{(x-2)\cdot n^n}{(n+1)^{n+1}} \right|$$

$$= |x-2| \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^{n+1}} \right| \leftarrow (n+1)^{n+1} = (n+1)^n (n+1)$$

$$= |x-2| \lim_{n\to\infty} \frac{n^n}{(n+1)^n (n+1)} \begin{array}{l} \to \infty \\ \to \infty \end{array}$$

$$= |x-2| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}$$

$$= |x-2|\, e^{\lim_{n\to\infty} \ln\left(\left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1}\right)}$$

$$= |x-2|\, e^{\lim_{n\to\infty} \left[\ln\left[\left(\frac{n}{n+1}\right)^n\right] + \ln\left(\frac{1}{n+1}\right)\right]}$$

$$= |x-2|\, e^{\lim_{n\to\infty} n \ln\left(\frac{n}{n+1}\right)} \cdot e^{\lim_{n\to\infty} \ln\left(\frac{1}{n+1}\right)}$$

$$= |x-2|\, e^{\lim_{n\to\infty} \frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}} \cdot \lim_{n\to\infty} \frac{1}{n+1} \to 0$$

$$\overset{H}{=} |x-2|\, e^{\lim_{n\to\infty} \frac{\frac{n+1}{n}\cdot\frac{1(n+1)-1(n)}{(n+1)^2}}{-\frac{1}{n^2}}} \cdot 0$$

$$= |x-2|\, e^{\lim_{n\to\infty}\left(-\frac{n^2}{1}\cdot\frac{n+1}{n}\cdot\frac{1}{(n+1)^2}\right)} \to -1$$

$$= |x-2|\, e^{-1} \cdot 0$$

$$= 0 < 1.$$

The series converges $(-\infty, \infty)$

$R = \infty$.

Example 6: Find the Taylor series for $f(x) = \cos x$ centered at $x = \pi$. Graph f and T_4 together.

T_4 = 4th degree Taylor polynomial.

fcts	@ $x=\pi$
$f(x) = \cos x$	-1
$f' = -\sin x$	0
$f'' = -\cos x$	1

$$\cos x = -1 + 0(x-\pi) + \frac{1}{2!}(x-\pi)^2 + \frac{0}{3!}(x-\pi)^3 - \frac{1}{4!}(x-\pi)^4 + \ldots$$

$$T_4(x) = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4$$

f	$= -\sin x$	0
f''	$= -\cos x$	1
$f^{(3)}$	$= \sin x$	0
$f^{(4)}$	$= \cos x$	-1
$f^{(5)}$	$= -\sin x$	0

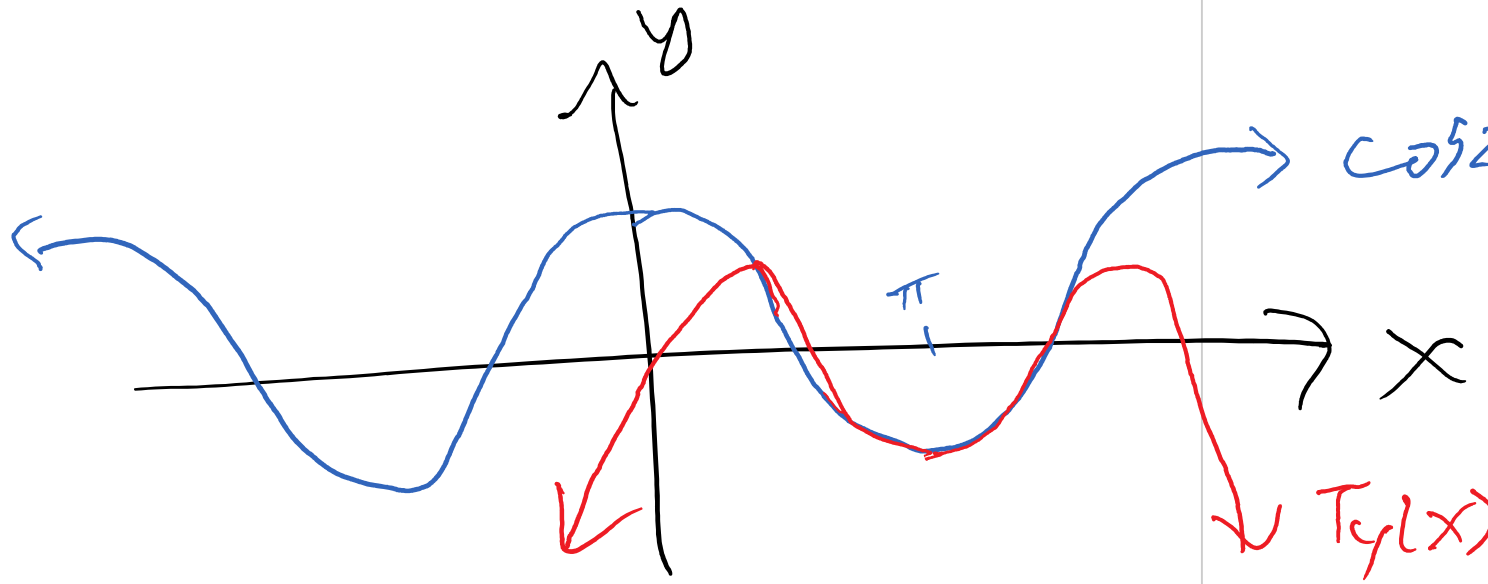
$+ \frac{0}{3!}(x-\pi)^3 - \frac{1}{4!}(x-\pi)^4 + \ldots$

$T_4(x) = -1 + \frac{1}{2}(x$ ... 24

so

$$\cos x = -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 - \frac{1}{8!}(x-\pi)^8 + \ldots$$

$f(x)$



Question: Why would it be beneficial to center the series at $x = \pi$?

if you are interested in values of cos x near $x = \pi$.

ex: $\cos(3)$, $\cos(2.9)$, $\cos(3.14)$

Example (8.4): Determine the number of terms of the Maclaurin Series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.

Recall: $|f^{(n+1)}(x)| = e^x$ $a = 0$

The Remainder Estimation Theorem: Suppose there exists a number M such that $|f^{(n+1)}(x)| \le M$ for all x in the interval $[a-c, a+c]$. The remainder of the nth-degree Taylor Polynomial for f centered at a satisfies: $|R_n(x)| \le M\frac{|x-a|^{n+1}}{(n+1)!}$

$\Rightarrow \underbrace{|R_n(0.1)|}_{\text{error}} \le M \frac{(0.1)^{n+1}}{(n+1)!} \le 0.00001$

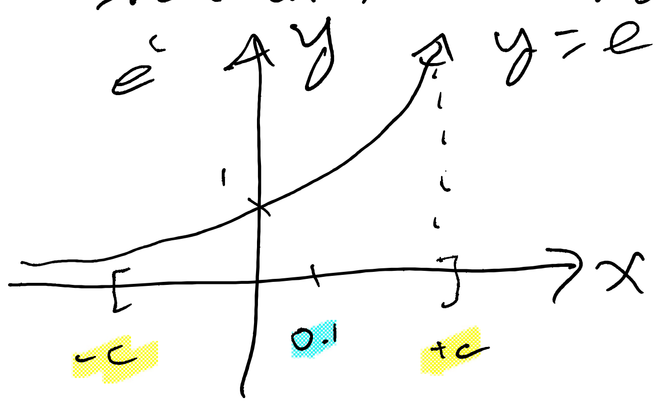
$e^x \le M$ on $[-c, c]$

want this less than 0.00001

Solve: $M \frac{1}{10^{n+1}(n+1)!} \le 0.00001$

2 unknowns n & M

We pick c so that $-c \le 0.1 \le +c$ such that we know e^c w/o a calculator.



choose $c = \ln 2$.

since I know $e^{\ln 2} = 2$

That is $e^x \le 2$ when

$-\ln 2 \le x \le +\ln 2$

My M

Solve $\frac{2}{10^{n+1}(n+1)!} \le 0.00001$

If $n=1$: $\frac{2}{100(2)} \le 0.00001$ False.

If $n=2$: $\frac{2}{1000(3!)} \le 0.00001$ False.

If $n=3$: $\underbrace{\frac{2}{10000(4!)}}_{0.00000 8\overline{3}} \le 0.00001$ True!

Conclusion: (3 terms)

$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}} + \cdots$

w/in "0.00001" of "real value".

❖ **Bonus Questions**

Example (9.1): Find the first three non-zero terms in the Maclaurin series for $f(x) = e^x \sin x$.

w/in 0.0001 of

the actual value

when $x = 0.1$.

check:

compare: $e^{0.1} = 1.105170918$

w/

approx: $1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{6}$

$= 1.10516\overline{6}$