15.2: Double Integrals over General Regions

Tuesday, December 6, 2022 9:55 AM

Double Integrals over General Regions

❖ **Double Integrals over General Regions**

Before we get into the "why" let's do the following example to get a taste for the "how":

__Example 1__: Evaluate:

$$\int_0^{\pi}\int_0^{\cos y}(x\sin y)\,dx\,dy = \int_0^{\pi}\left[\frac{x^2}{2}\sin y\right]_{x=0}^{x=\cos y} dy$$

$$= \int_0^{\pi}\frac{1}{2}\cos^2 y\sin y\,dy$$

$$= -\frac{1}{2}\cdot\frac{1}{3}\cos^3 y\Big]_{y=0}^{y=\pi}$$

$$= -\frac{1}{6}\left((-1)^3-(1)^3\right)$$

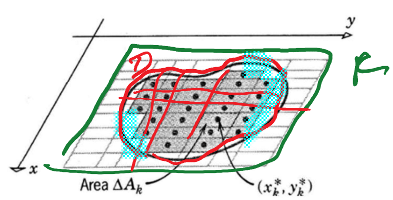
$$= \frac{1}{3}$$

In the last section we saw how to use double integrals to find the volume bounded between a multi-variable function and a rectangular region R on the xy-plane.

In this section we will look at more general regions that are not rectangles. We will label these more general regions D.

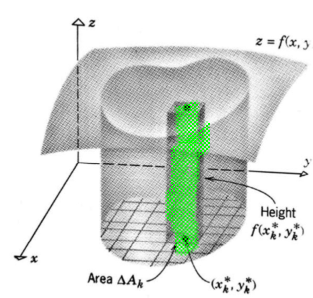
To do this we will suppose that D is a bounded region and can be enclosed in a rectangular region R.

manipulate 16.09

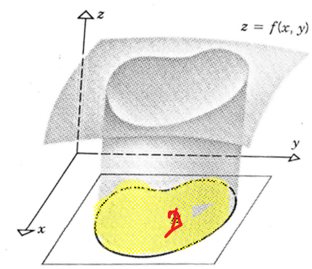


From here, the process is very similar. We will divide D into small rectangles (note that on the edge we have non-rectangular shapes which is okay since we are approximating). We then find the volume of each rectangular cylinder and add them to each other.

manipulate 16.10

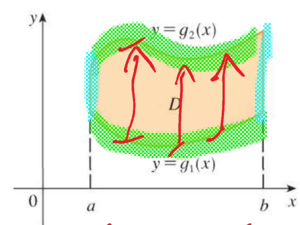


To find a better approximation we will increase the number of the rectangles to infinity which takes us to the double integral definition of volume: $V = \iint\limits_D f(x,y)\,dA$



Plane regions can be extremely complex, and the theory of double integrals over very general region is a topic for advanced courses in mathematics. We will limit our study of double integrals to two basic types of regions:

A **Type** I **region** is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by the continuous curves $y = g_1(x)$ and $y = g_2(x)$. Then:



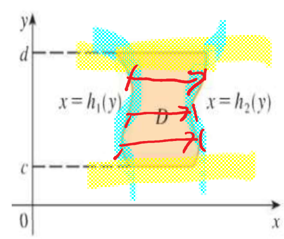
$$\iint\limits_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Arrows are parallel to y-axis

where $D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$

Since x is fixed for the first integration, draw a vertical line in the region. The lower point of the intersection is the curve $y = g_1(x)$ and the higher point is $y = g_2(x)$. These are the lower and upper y-limits of integration.

A **Type** II **region** is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by the continuous curves $x = h_1(y)$ and $x = h_2(y)$. Then:



$$\iint\limits_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

where $D = \{(x,y) \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$

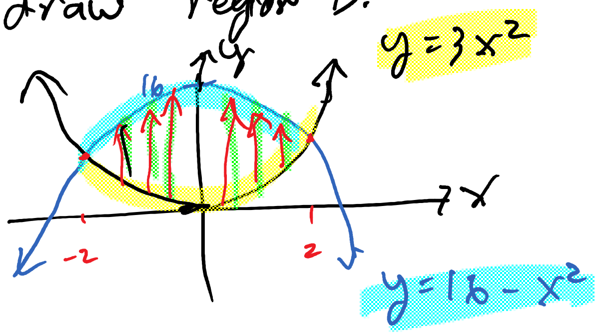
Arrows parallel to x-axis

Since y is fixed for the first integration, draw a horizontal line in the region. The leftmost point of the intersection is the curve $x = h_1(y)$ and the rightmost point is $x = h_2(y)$. These are the lower and upper x-limits of integration.

Example 2: Evaluate $I = \iint_D 2x^2 y\,dA$ over the region enclosed between $y=3x^2$, $y=16-x^2$, $x=-2$ and $x=2$.

Manipulate 16.13 and 16.14

① draw region D.



solve $3x^2 = 16 - x^2$

$\Rightarrow 4x^2 = 16$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

② set up iterated integral.

$$I = \int_{-2}^{2}\int_{y=3x^2}^{y=16-x^2} 2x^2 y \, dy\, dx$$

$$= \int_{-2}^{2} \left[x^2 y^2 \right]_{y=3x^2}^{y=16-x^2} dx$$

$$= \int_{-2}^{2} x^2\Big[\underbrace{(16-x^2)^2}_{256-32x^2+x^4} - \underbrace{(3x^2)^2}_{9x^4}\Big]\, dx$$

$$= \int_{-2}^{2} (256x^2 - 32x^4 - 8x^6)\, dx$$

$$= 2\int_{0}^{2} \left[256x^2 - 32x^4 - 8x^6\right] dx$$

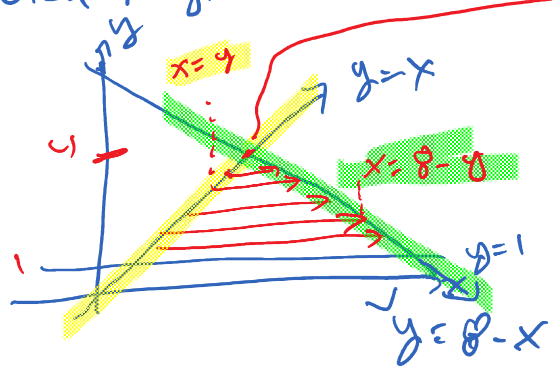
$$= 2\left[\frac{256}{3}x^3 - \frac{32}{5}x^5 - \frac{8}{7}x^7\right]_0^2$$

$$= \frac{69632}{105}$$

Example 3: Evaluate $\iint\limits_D \left(2+\frac{1}{y}\right) dA$ over the region enclosed between $y = 8 - x$, $y = x$ and $y = 1$.

① sketch region

where do they intersect?



Manipulate
16.16 and 16.17

$y = 8 - y$

$2y = 8$

$y = 4$

② set up and evaluate iterated integral.

$$\iint\limits_D \left(2+\frac{1}{y}\right) dA = \int_{y=1}^{y=4}\int_{x=y}^{x=8-y} \left(2+\frac{1}{y}\right) dx\, dy$$

$$= \int_1^4 \Big[x\left(2+\frac{1}{y}\right)\Big]_{x=y}^{x=8-y} dy$$

$$= \int_1^4 \left(2+\frac{1}{y}\right)\underbrace{(8-y-y)}_{8-2y} dy$$

$$= \int_1^4 \left(16 - 4y + \frac{8}{y} - 2\right) dy$$

14

$$= 14y - 2y^2 + 8\ln|y| \Big]_1^4$$

$$= 56 - 32 + 8\ln 4 - 14 + 2$$

$$= 12 + 8\ln(4).$$

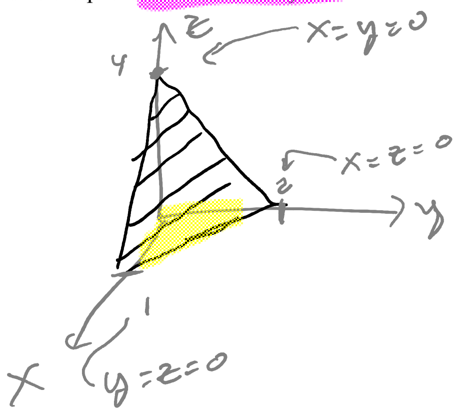
$= 12 + 8\ln(4).$

Historical Note: Our definition of the definite integral is based upon the work of the German mathematician Bernhard Riemann (1826 – 1866). The son of a pastor, Riemann's plan was to become a minister like his father. However, he was too good at math to set it aside. So when he arrived at university, and with his family's permission, he began to study under Carl Gauss. His work as one of the key contributors to the rigorization of mathematics remains very influential.

Personally, he was shy and suffered from numerous nervous breakdowns. He had a terrible fear of public speaking. He was also a perfectionist who wouldn't publish anything unless he felt it was perfect. The story goes that some of his unpublished papers were discarded by a maid who did not realize their potential value. He died in Italy having had to flee war in Germany.

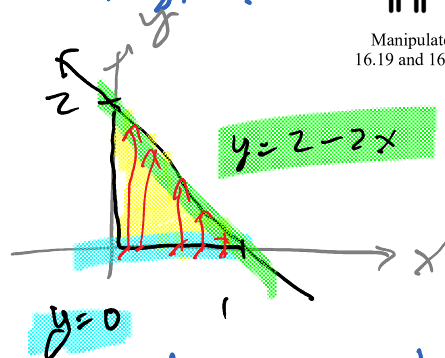
While he did not pursue the ministry, Riemann remained dedicated to his faith for his lifetime. He saw his work as a mathematician as another way to serve God and considered his faith the most important aspect of his life. His tombstone ends with the inscription, "For those who love God, all things must work together for the best."

Example 4: Use a double integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $z = 4 - 4x - 2y$.



① sketch the region!

Manipulate
16.19 and 16.20



② set up and evaluate iterated integral.

$$\text{volume} = \int_{x=0}^{x=1}\int_{y=0}^{y=2-2x} \boxed{4-4x-2y}\,dy\,dx$$

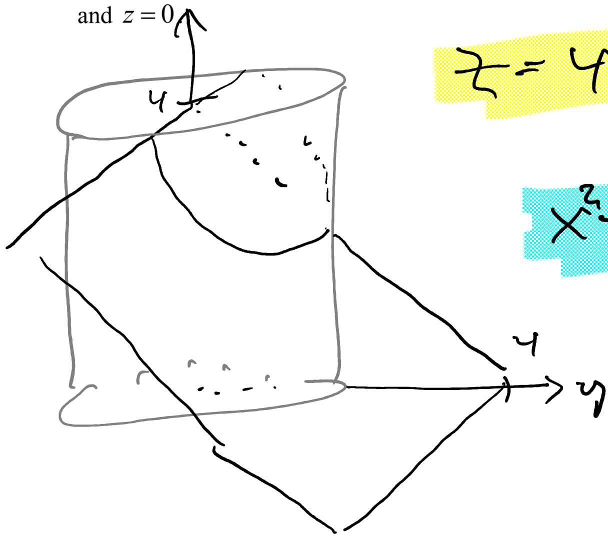
$$= \int_0^1 \Big[4y - 4xy - y^2\Big]_{y=0}^{y=2-2x} dx$$

$$= \int_0^1 \Big[\underbrace{4(2-2x) - 4x(2-2x) - (2-2x)^2}_{8-8x-8x+8x^2-4+8x-4x^2}\Big]\, dx$$

$$= \int_0^1 (4 - 8x + 4x^2)\,dx$$

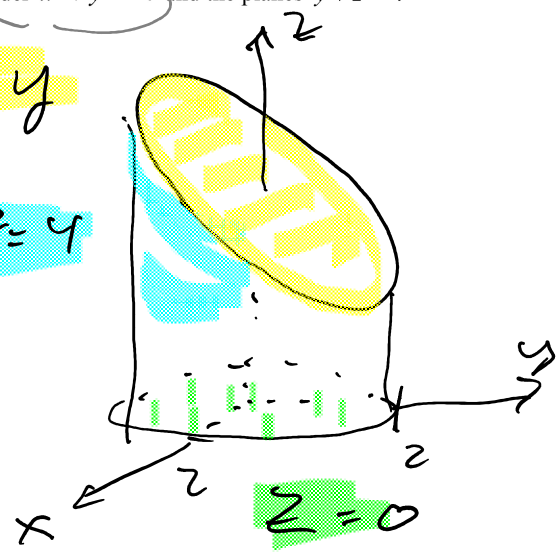
$$= \Big[4x - 4x^2 + \tfrac{4}{3}x^3\Big]_0^1 = \frac{4}{3}$$

Example 5: Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.



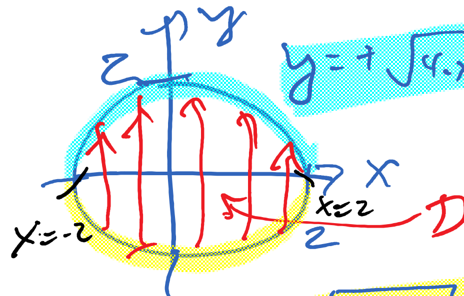
$z = 4 - y$

$x^2 + y^2 = 4$



$z = 0$

① sketch the region



$y = +\sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

$$x^2 + y^2 = 4$$

$$\Rightarrow y^2 = 4 - x^2$$

$$\Rightarrow y = \pm\sqrt{4 - x^2}$$

② set-up and evaluate an iterated integral!

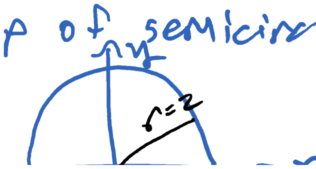
$$\iint_D (4-y)\, dA = \int_{-2}^{2} \int_{y=-\sqrt{4-x^2}}^{y=+\sqrt{4-x^2}} 4 - y \; dy\, dx$$

$$= \int_{-2}^{2} \Big[4y - \tfrac{1}{2}y^2 \Big]_{y=-\sqrt{4-x^2}}^{y=+\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \Big[\Big(4\sqrt{4-x^2} - \cancel{\tfrac{1}{2}(4-x^2)}\Big) - \Big(-4\sqrt{4-x^2} - \cancel{\tfrac{1}{2}(4-x^2)}\Big) \Big] dx$$

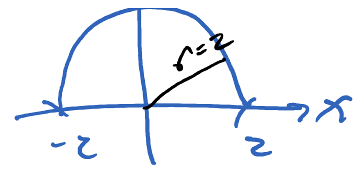
$$= \int_{-2}^{2} 8\sqrt{4-x^2}\, dx$$

Top of semicircle



$$= 8 \int_{-2}^{2} \sqrt{4-x^2}\, dx$$

$$= 8\underbrace{\int_{-2}^{2}\sqrt{4-x^2}\,dx}_{2\pi}$$



$$\text{Area} = \frac{\pi\cdot 2^2}{2} = 2\pi$$

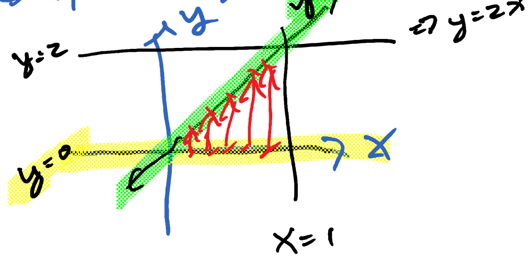
$$= 16\pi.$$

Example 6: Evaluate $\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy$

$\frac{y}{2} \le x \le 1$

$0 \le y \le 2$

① sketch region. $y=2x$ $\quad x=\frac{y}{2} \Rightarrow y=2x$



check ✓

② setup and evaluate iterated integral.

$$\int_0^2 \int_{y/2}^1 e^{x^2}\,dx\,dy = \int_0^1 \int_{y=0}^{y=2x} e^{x^2}\,dy\,dx$$

$$= \int_0^1 \left[y e^{x^2} \right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 2x e^{x^2}\,dx$$

$$= e^{x^2}\Big|_0^1$$

$$= e-1.$$

❖ Properties of Double Integrals

The following properties can be proven just like we did in ~~Math 152~~ a previous calc class.

- $\iint\limits_D [f(x,y)+g(x,y)]\,dA = \iint\limits_D f(x,y)\,dA + \iint\limits_D g(x,y)\,dA$
- $\iint\limits_D c\,f(x,y)\,dA = c\iint\limits_D f(x,y)\,dA$ where c is constant

Although double integrals arose in the context of calculating volumes, they can also be used to calculate areas. For this purpose, we consider the solid consisting of the points between the plane $z=1$ and the region D, in the *xy*-plane. The volume of this solid is

$$V = \iint\limits_D 1\,dA = \iint\limits_D dA$$

Manipulate
16.25

However, the solid has volume:

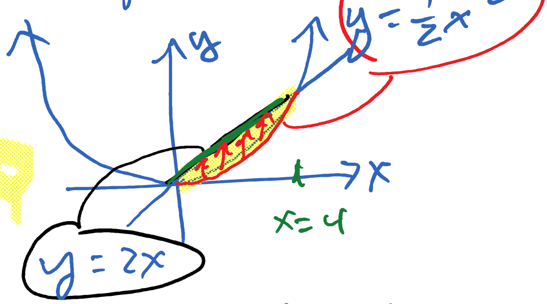
$$V = (\text{area of base})(\text{height}) = (\text{area of } D)(1) = (\text{area of } D)$$

Therefore:

$$\text{area of } D = \iint\limits_D dA$$

Example 7: Use a double integral to find the area of the region enclosed between the parabola $y = \frac{1}{2}x^2$ and the line $y = 2x$.

① pictures:



$2x = \frac{1}{2}x^2$
$\Rightarrow 4x = x^2$
$\Rightarrow 0 = x(x-4)$

② set-up and evaluate iterated integral!

$$\text{Area} = \int_0^4 \int_{y=\frac{1}{2}x^2}^{y=2x} 1\,dy\,dx$$

$$= \int_0^4 \left[y\right]_{y=\frac{1}{2}x^2}^{y=2x} dx$$

$$= \int_0^4 \left(2x - \tfrac{1}{2}x^2\right) dx$$

$$= \left[\tfrac{2}{2}x^2 - \tfrac{1}{6}x^3\right]_0^4$$

$$= 16 - \cancel{\tfrac{64}{6}}\ \tfrac{32}{3}$$

$= 16 - \cancel{\frac{64}{6}}\ \frac{32}{3}$

$= \frac{16}{3}$