14.3: Partial Derivatives

Friday, November 11, 2022 7:55 AM

Partial Derivatives

The process of finding the derivative of a function with respect to one independent variable while holding the other variables constant is called **partial differentiation**. Each derivative is called a **partial derivative**.

Note: The notation of partial derivatives that we are about to learn looks intimidating. But this is actually very straight-forward and no more difficult than ordinary derivatives. Similarly, if you understand derivatives graphically, you should have no trouble understanding the graphical interpretation of partial derivatives.

❖ **Partial Derivatives**

Functions of several variables have several derivatives, one for each variable. In the following example, we find these **partial derivatives** without (yet) having them clearly defined.

Example 1: If $f(x,y)=2x^3y^2+2y+4x$ find the following.

a) $f_x(x,y)$ ← Treat y like a constant.

$$f(x,y)=2x^3y^2+2y+4x$$

$$\Rightarrow f_x(x,y)=2\cdot 3x^2y^2+0+4(1)$$

$$=6x^2y^2+4$$

b) $f_y(1,2)$ ← Treat x like a constant. Then evaluate the **partial derivative** when $(x,y)=(1,2)$

$$f(x,y)=2x^3y^2+2y+4x$$

$$\Rightarrow f_y(x,y)=2x^3\cdot 2y+2(1)+4(0)$$

$$=4x^3y+2$$

And $f_y(1,2)=4(1)^3(2)+2=10.$

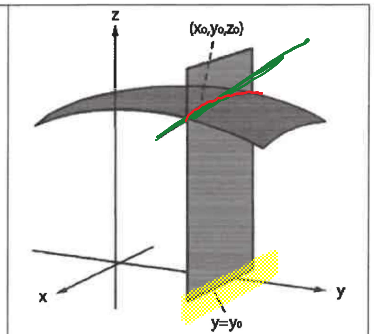
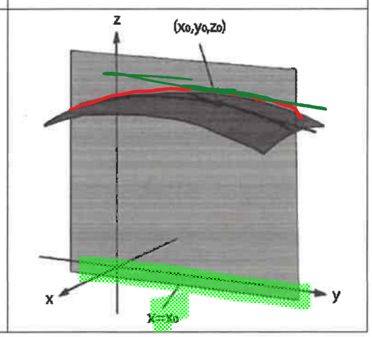
❖ Interpretation of Partial Derivatives

The graph of $z = f(x, y)$ is a surface in space.

Short version: f_x is the slope in the x-direction and f_y is the slope in the y-direction.

Long version:

Manipulate 15.31

If the variable y is fixed at $y = y_0$, then $z = f(x, y_0)$ is a function of one variable. The graph of this function is the curve of intersection between $z = f(x, y)$ and plane $y = y_0$. $f_x(x, y_0)$ represents the slope of tangent line to this curve. Which means **the slope of the surface** $z = f(x, y)$ in x-**direction**. Or **rate of change of** z **in direction of** x.	
Similarly, $f_y(x_0, y)$ represents the **slope of the surface** $z = f(x, y)$ **in** y**-direction.** Or **rate of change of** z **in direction of** y.	

Manipulate 15.32

Example 1b revisited: If $f(x, y) = 2x^3y^2 + 2y + 4x$ interpret $f_y(1, 2)$.

recall that $f_y(1, 2) = 10$.

Hold x constant at $x = 1$ and look at the $\boxed{\text{curve}}$ where the plane $x = 1$ and the surface $f(x, y)$ intersect. The slope of that $\boxed{\text{curve}}$ in the direction parallel to the y-axis is 10 at the point $(x, y) = (1, 2)$.

Alt. explanation

$f(1, 3)$ vs $f(1, 2)$

about 10 more than

$f(1, 1)$ vs $f(1, 2)$

about 10 less than

❖ **Notation for first partial derivatives**

won't use much.

- First partial derivatives of $z = f(x, y)$:
  - with respect to x:
    - $\frac{\partial z}{\partial x} = z_x = f_x(x,y) = f_x = \frac{\partial}{\partial x} f(x,y) = \frac{\partial f}{\partial x} = D_x f = D_1 f = f_1$

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  - with respect to y:
    - $\frac{\partial z}{\partial y} = z_y = f_y(x,y) = f_y = \frac{\partial}{\partial y} f(x,y) = \frac{\partial f}{\partial y} = D_y f = D_2 f = f_2$
- The values of the first partial derivatives at the point (a, b) are detonated by
  - $\left.\frac{\partial z}{\partial x}\right|_{(a,b)} = f_x(a,b) = \frac{\partial}{\partial x} f(a,b)$
  - $\left.\frac{\partial z}{\partial y}\right|_{(a,b)} = f_y(a,b) = \frac{\partial}{\partial y} f(a,b)$

To calculate/evaluate an expression such as $f_x(0,2)$, first differentiate with respect to x, and then evaluate the resulting expression at $(0,2)$.

The concept of a partial derivative can be extended to functions of 3 or more variables. The function $w = f(x, y, z)$ has 3 partial derivatives, each of which can be found by holding 2 of the variables constant.

Example 2: If $z = x^4 \sin(xy^3)$ find the following.

a) $\frac{\partial z}{\partial x} = 4x^3 \sin(xy^3) + x^4 \cdot \cos(xy^3) \cdot y^3$

slopes in the x direction

Held y constant

b) $\frac{\partial z}{\partial y} = x^4 \cos(xy^3) \cdot x \cdot 3y^2$

Held x constant.

slopes in the y direction

❖ *Higher-Order Partial Derivatives*

You can differentiate a function more than once to find partial derivatives of second, third or higher order.
Second Partial Derivatives of $z = f(x,y)$:

Leibniz notation
read: R → L

- Twice with respect to x: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$
- Twice with respect to y: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$
- First with respect to x and then with respect to y: $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$
- First with respect to y and then with respect to x: $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$

f notation
read: L → R

The last two cases are called **mixed partial derivatives**.

Caution: Notice the order in which x and y are listed in the mixed partials in the two notations. In Leibniz notation, the order reminds us of process where as with subscripts the order matches the order of the partials.

Partial derivative of order three or higher can also be defined. For example:

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x}.$$

L → R

Example 3: Find the second-order partial derivatives of $f(x,y) = x^2y^3 + x^4y$.

$f_x(x,y) = 2xy^3 + 4x^3y$

$f_{xx}(x,y) = 2y^3 + 12x^2y$

$f_{xy} = 6xy^2 + 4x^3$

$f_y = 3x^2y^2 + x^4$

$\frac{\partial^2 f}{\partial y^2} = 6x^2y$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = 6xy^2 + 4x^3$

$= f_{yx}$

Clairaut's Theorem: Assuming f is defined on an open set D of $\mathbb{R}^2$, and that f_{xy} and f_{yx} are continuous throughout D. Then $f_{xy} = f_{yx}$ at all points of D. mixed partials are equal.

Example 4: Let $f(x,y) = y^2e^x + y$. Find f_{xyy}.

$$f_x = y^2e^x + 0 = y^2e^x$$

$$f_{xy} = 2ye^x$$

$$f_{xyy} = 2e^x. \leftarrow f_{xyy}(x,y).$$

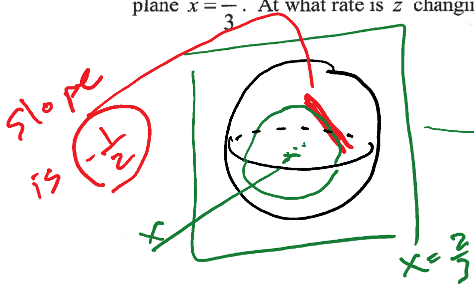
Example 5: If $f(x,y,z) = x^2\cos z \sin y$, find $\dfrac{\partial^3 f}{\partial y \partial y \partial x}$.

$$\frac{\partial f}{\partial x} = 2x\cos z \sin y.$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = 2x\cos z \cdot \cos y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \frac{\partial^3 f}{\partial y^2 \partial x} = -2x\cos z \cdot \sin y$$

Example 6: Suppose that a point Q moves along the intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x = \frac{2}{3}$. At what rate is z changing with respect to y when the point is at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$?



$$x^2 + y^2 + z^2 = 1$$

$$z = +\sqrt{1 - x^2 - y^2}$$

we want: z_y

$$z_y = \frac{1}{\cancel{2}\sqrt{1-x^2-y^2}} \cdot (-\cancel{2}y)$$

$$= \frac{-y}{\sqrt{1-x^2-y^2}}\bigg|_{\left(\frac{2}{3},\frac{1}{3}\right)}$$

$$z_y = \frac{-1/3}{\sqrt{1-\frac{4}{9}-\frac{1}{9}}} = \frac{-1/3}{2/3} = -1/2$$

z is changing at a rate of "$-\frac{1}{2}$"