13.2: Derivatives and Integrals of Vector Functions

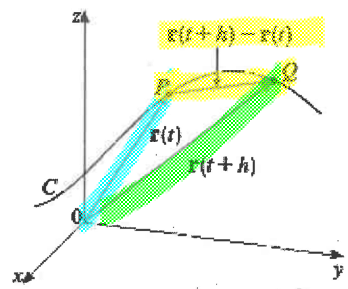
Saturday, October 15, 2022 11:27 AM

Derivatives and Integrals of Vector Functions

❖ **Derivatives**

The **derivative** $\vec{r}\,'$ of a vector valued function $\vec{r}$ is defined in much the same was as for real-valued functions: $\dfrac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim\limits_{h\to 0}\dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$ if this limit exists.

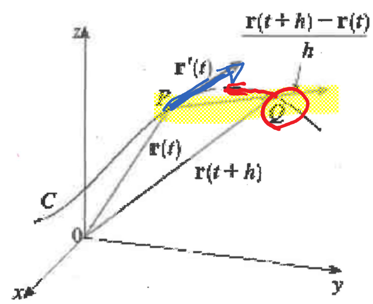
The geometric significance of this definition is shown in the diagrams to the right. If the points P and Q have position vectors $\vec{r}(t)$ and $\vec{r}(t+h)$, then $\overline{PQ}$ represents the vector $\vec{r}(t+h)-\vec{r}(t)$ which can therefore be regarded as a secant vector.



(a) The secant vector $\overrightarrow{PQ}$

manipulate
14,08

If $h>0$, the scalar multiple $\dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$ has the same direction and as $h\to 0$, it appears that this vector approaches a vector that lies on the tangent line. For this reason, the vector $\vec{r}\,'(t)$ is called the **tangent vector** to the curve (provided it exists and is non-zero). This is shown to the right.



(b) The tangent vector $\mathbf{r}'(t)$

We will also have occasion to consider the **unit tangent vector** which is $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

manipulate
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The following theorem gives us a convenient method for computing the derivative of a vector function $\vec{r}(t)$; just differentiate each component of $\vec{r}(t)$.

<u>**Theorem**</u>: If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ where f, g, and h are differentiable functions, then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle$

Example 1: Find the velocity, speed and acceleration of a particle whose motion in space is given by the position vector $\vec{r}(t) = 2\cos t\vec{i} + 2\sin t\vec{j} + 5\cos^2 t\vec{k}$.

Velocity: $\vec{v}(t) = \vec{r}'(t) = \langle -2\sin t, 2\cos t, \underbrace{-10\cos t\sin t}_{-5\sin 2t} \rangle$

speed: $s(t) = |\vec{v}(t)| = |\vec{r}'(t)|$

$$= \sqrt{4\sin^2 t + 4\cos^2 t + 100\cos^2 t\sin^2 t}$$

$$= \sqrt{4 + 100\cos^2 t\sin^2 t} \neq 2 + 10\cos t\sin t.$$

acceleration: $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$

$$= \langle -2\cos t, -2\sin t, -10\cos 2t \rangle$$

Example 2: Find the unit tangent vector of the curve $\vec{r}(t) = (3\cos t)\vec{i} + (3\sin t)\vec{j} + t^2\vec{k}$

$\vec{T}(t) = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|} \longleftarrow$ Memorize.

need $\vec{r}'(t) = \langle -3\sin t, 3\cos t, 2t \rangle$

and $|\vec{r}'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$

$$= \sqrt{9 + 4t^2} \neq 3 + 2t^2$$

Thus: $\vec{T}(t) = \dfrac{1}{\sqrt{9+4t^2}} \langle -3\sin t, 3\cos t, 2t \rangle$

Theorem Suppose **u** and **v** are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

All work as you might expect.

❖ **Integrals**

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}$$

This means we can evaluate an integral of a vector function by integrating each component function. We can also extend the Fundamental Theorem of Calculus to continuous ~~factor~~ vector functions as follows:

$$\int_a^b \vec{r}(t)\,dt = \left[R(t)\right]_a^b = \vec{R}(b) - \vec{R}(a)$$

Example 3: Suppose $\vec{r}(t) = (\cos t)\vec{i} + \vec{j} - (2t)\vec{k}$. Evaluate the following integrals:

a) $\int \vec{r}(t)dt = \left\langle \int \cos t\,dt, \int 1\,dt, \int -2t\,dt \right\rangle$

$= \langle \sin t + C_1, t + C_2, -t^2 + C_3 \rangle$

$= \langle \sin t, t, -t^2 \rangle + \vec{C}$

b) $\int_0^{\pi} \vec{r}(t)\,dt = \left[\langle \sin t, t, -t^2 \rangle + \vec{C}\right]_0^{\pi}$

$= \left(\langle 0, \pi, -\pi^2 \rangle + \vec{C}\right) - \left(\langle 0, 0, 0 \rangle + \vec{C}\right)$

$= \langle 0, \pi, -\pi^2 \rangle$

Example 4: Suppose we don't know the path of a hang glider, but only its acceleration vector $\vec{a}(t) = -(3\cos t)\vec{i} - (3\sin t)\vec{j} + 2\vec{k}$. We also know that at take-off (where $t = 0$) the glider departed from the point $(3,5,0)$ with velocity $\vec{v}(0) = 4\vec{j}$. Find the glider's position as a function of t.

$\vec{a}(t) \longrightarrow \vec{v} = \int \vec{a}(t)\,dt \longrightarrow \vec{r} = \int \vec{v}(t)\,dt$.

Find $\vec{v}$: $\vec{v}(t) = \int \langle -3\cos t, -3\sin t, 2\rangle\,dt$

$= \langle -3\sin t, 3\cos t, 2t\rangle + \vec{C}$

including its constant: $\vec{v}(0) = \langle 0, 3, 0\rangle + \vec{C} = \langle 0, 4, 0\rangle$

$\vec{C} = \langle 0, 1, 0\rangle$

Find $\vec{r}$: $\vec{r}(t) = \int \langle -3\sin t, 3\cos t + 1, 2t\rangle\,dt$

$= \langle 3\cos t, 3\sin t + t, t^2\rangle + \vec{D}$

including its constant: $\vec{r}(0) = \langle 3, 0, 0\rangle + \vec{D} = \langle 3, 5, 0\rangle$

Thus: $\vec{r}(t) = \langle 3\cos t, 3\sin t + t + 5, t^2\rangle$