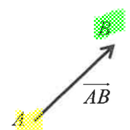
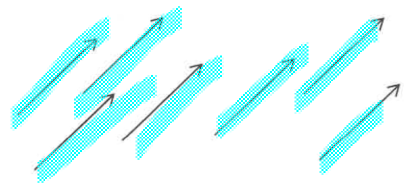
Vectors

Some quantities such as length, area and temperature are determined by a number, which is called the *magnitude*. Such quantities are called **scalar**. On the other hand, to describe the displacement of an object, two numbers are required: the *magnitude* and the *direction*. Similarly, while describing the velocity of a moving object we discuss the speed and direction of travel. Quantities such as displacement, velocity, acceleration and force that involve *magnitudes* as well as *direction* are called **direct quantities**. One way to represent such quantities mathematically is through the use of **vectors**.

A **vector** in the plane is a line segment (has magnitude-length) with an assigned direction. The vector $\overrightarrow{AB}$ as shown has initial point A and end point B. Its magnitude is denoted by $\left|\overrightarrow{AB}\right|$.





Two vectors are considered **equal** if their magnitude and direction are the same. All the vectors shown above are the *same*! To understand this better, think about a vector as a displacement. No matter how I get from one point (initial point) to the other (endpoint) or where those points are, my displacement is the same.

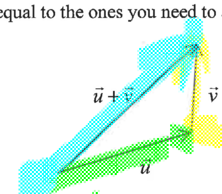
❖ Sum of Vectors

> **Definition of Vector Addition** If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

bold font $\vec{u}$ or $\underset{=}{u}$

To find the sum of two vectors geometrically, do one of the following:

a) Triangle Law: Sketch vectors equal to the ones you need to add, with the initial point of one at the terminal point of the other.



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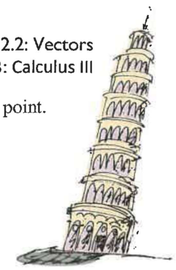
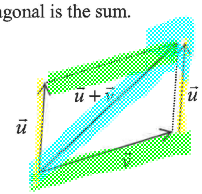
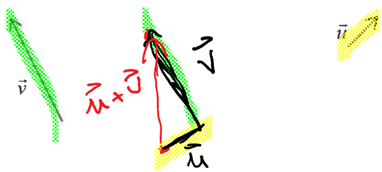


b) Parallelogram Law: Sketch vectors equal to the ones you need to add, starting at the same point. Create a parallelogram and its diagonal is the sum.



manipulate 13.08 and 13.14

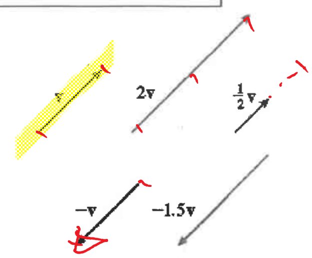
Example 1: Sketch $\vec{u}+\vec{v}$.



❖ **Multiplication of Vectors by a Scalar**

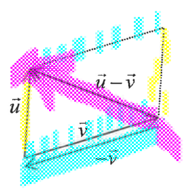
> **Definition of Scalar Multiplication** If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.

manipulate 13.05



❖ **Difference of Vectors**

The difference of two vectors $\vec{u}$ and $\vec{v}$ is defined by $\vec{u}-\vec{v}=\vec{u}+(-\vec{v})$ where $-\vec{v}=(-1)\vec{v}$ is the vector that has the same length as $\vec{v}$ but opposite direction. To find the difference geometrically use the parallelogram.



manipulate 13.09

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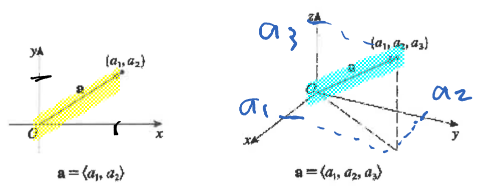
Example 2: Sketch $2\vec{u}-\vec{v}$.

❖ **Components**

For some purposes it's best to introduce a coordinate system and treat vectors algebraically. If we place the initial point of a vector **a** at the origin of a rectangular coordinate system, then the terminal point of **a** has coordinates of the form (a_1, a_2) or (a_1, a_2, a_3), depending on whether our coordinate system is two- or three-dimensional (see Figure 11`



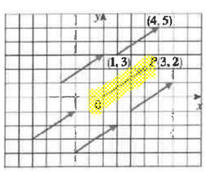


FIGURE 12
Representations of the vector $\mathbf{a} = \langle 3, 2 \rangle$

These coordinates are called the **components** of **a** and we write

$$\mathbf{a} = \langle a_1, a_2 \rangle \quad \text{or} \quad \mathbf{a} = \langle a_1, a_2, a_3 \rangle$$

We use the notation $\langle a_1, a_2 \rangle$ for the ordered pair that refers to a vector so as not to confuse it with the ordered pair (a_1, a_2) that refers to a point in the plane.

For instance, the vectors shown in Figure 12 are all equivalent to the vector $\overrightarrow{OP} = \langle 3, 2 \rangle$ whose terminal point is $P(3, 2)$. What they have in common is that the terminal point is reached from the initial point by a displacement of three units to the right and two upward. We can think of all these geometric vectors as **representations** of the algebraic vector $\mathbf{a} = \langle 3, 2 \rangle$. The particular representation $\overrightarrow{OP}$ from the origin to the point $P(3, 2)$ is called the **position vector** of the point P.

In three dimensions, the vector $\mathbf{a} = \overrightarrow{OP} = \langle a_1, a_2, a_3 \rangle$ is the **position vector** of the point $P(a_1, a_2, a_3)$. (See Figure 13.) Let's consider any other representation $\overrightarrow{AB}$ of **a**, where the initial point is $A(x_1, y_1, z_1)$ and the terminal point is $B(x_2, y_2, z_2)$. Then we must have $x_1 + a_1 = x_2$, $y_1 + a_2 = y_2$, and $z_1 + a_3 = z_2$ and so $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$, and $a_3 = z_2 - z_1$. Thus we have the following result.

⟨ , ⟩ vs (,)
vector   point

Manipulate 13.12

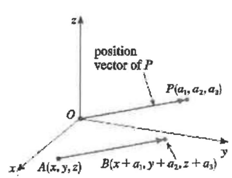


FIGURE 13
Representations of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$

1 Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 3: Find the vector represented by the directed line segment with initial point A $(2,4,-3)$ and terminal point B $(-1,2,1)$.

$\overrightarrow{AB} = \langle -1-2, 2-4, 1-(-3)\rangle$
$= \langle -3, -2, 4\rangle$

The **magnitude** or **length** of the vector **v** (also written $\vec{v}$) is the length of any of its representations and is denoted by the symbol $|\vec{v}|$ or $\|\vec{v}\|$. By using the distance formula to compute the length of a segment *OP*, we obtain the following formulas.

not bold

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2\rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

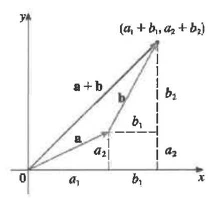
★ manipulate 13.40

To add, subtract, or find scalar multiples of a vector, perform the operations component-wise.

If $\mathbf{a} = \langle a_1, a_2\rangle$ and $\mathbf{b} = \langle b_1, b_2\rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2\rangle \qquad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2\rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2\rangle$$



Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3\rangle + \langle b_1, b_2, b_3\rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3\rangle$$
$$\langle a_1, a_2, a_3\rangle - \langle b_1, b_2, b_3\rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3\rangle$$
$$c\langle a_1, a_2, a_3\rangle = \langle ca_1, ca_2, ca_3\rangle$$

Example 4: If $\vec{u} = \langle 0,-3,4\rangle$ and $\vec{v} = \langle 2,-1,-2\rangle$ find the following:

a) $|\vec{u}| = \sqrt{0^2 + (-3)^2 + 4^2} = \sqrt{25} = 5$

b) $5\vec{v} = 5\langle 2,-1,-2\rangle = \langle 5(2), 5(-1), 5(-2)\rangle$
$= \langle 10, -5, -10\rangle$

c) $\vec{u}+\vec{v}$ $= \langle 0,-3,4\rangle + \langle 2,-1,-2\rangle$
$= \langle 0+2, -3+(-1), 4+(-2)\rangle$
$= \langle 2,-4,2\rangle$

d) $\vec{u}-\vec{v}$
$= \langle 0,-3,4\rangle - \langle 2,-1,-2\rangle$
$= \langle 0-2, -3-(-1), 4-(-2)\rangle = \langle -2,-2,6\rangle$

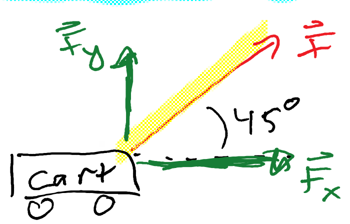
e) $3\vec{u}+2\vec{v}$
$= 3\langle 0,-3,4\rangle + 2\langle 2,-1,-2\rangle$
$= \langle 0,-9,12\rangle + \langle 4,-2,-4\rangle$
$= \langle 4,-11,8\rangle$

Example 5: Find the component form and length of the vector with initial point $P(-3,4,1)$ and terminal point $Q(-5,2,2)$.

$\overrightarrow{PQ} = \langle -5-(-3), 2-4, 2-1\rangle$
$= \langle -2,-2,1\rangle$

Example 6: A small cart is being pulled along a smooth horizontal floor with a 20-lb force $\vec{F}$ making a 45° angle to the floor. What is the effective force moving the cart forward?

Hint: the effective force is the horizontal component of $\vec{F}$



20lbs @ 45° from horizontal.

$\vec{F}_x + \vec{F}_y = \vec{F} = \langle a, b\rangle = \langle a,a\rangle$

same magnitude. $|\vec{F}_x| = |\vec{F}_y|$

$20 = |\langle a,a\rangle| = \sqrt{a^2+a^2}$
$= a\sqrt{2}$

$a = \frac{20}{\sqrt{2}}$

$\Rightarrow \vec{F}_x = \langle \frac{20}{\sqrt{2}}, 0\rangle$

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The effective force is $\frac{20}{\sqrt{2}}$ lbs in the horizontal direction.

We denote by V_2 the set of all two-dimensional vectors and by V_3 the set of all three-dimensional vectors. More generally, in subsequent math courses, we will need to consider the set V_n of all n-dimensional vectors. An n-dimensional vector is an ordered n-tuple: $\vec{a} = \langle a_1, a_2, \ldots, a_n \rangle$ where $a_1, a_2, \ldots, a_n$ are real numbers that are called the components of $\vec{a}$. Addition and scalar multiplication are defined in terms of components just as for the cases $n = 2$ and $n = 3$.

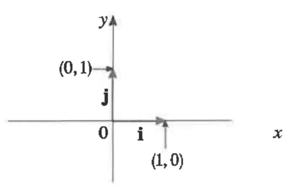
Properties of Vectors If **a**, **b**, and **c** are vectors in V_n and c and d are scalars, then

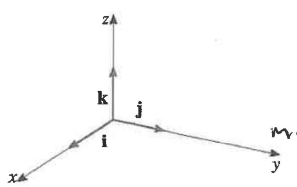
1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$	**2.** $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \vec{\mathbf{0}} = \mathbf{a}$	**4.** $\mathbf{a} + (-\mathbf{a}) = \vec{\mathbf{0}}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$	**6.** $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$	**8.** $1\mathbf{a} = \mathbf{a}$

Three vectors in V_3 play a special role. Let

$$\mathbf{i} = \langle 1, 0, 0 \rangle \qquad \mathbf{j} = \langle 0, 1, 0 \rangle \qquad \mathbf{k} = \langle 0, 0, 1 \rangle$$

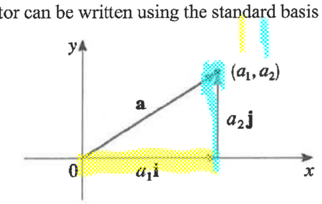
These vectors $\vec{i}, \vec{j}$, and $\vec{k}$ are called the **standard basis vectors**. They have length 1 and point in the directions of the positive x-, y-, and z-axes. Similarly, in two dimensions $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$.



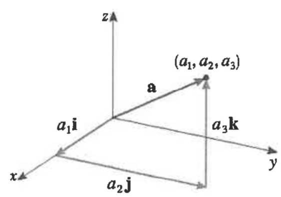


manipulate
13, 17

Any vector can be written using the standard basis vectors.



(a) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$



(b) $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

$\langle a_1, a_2, a_3 \rangle$

Example 7: Express the following vectors in term of standard basis vectors.

a) $\langle 5,-2,6\rangle = 5\vec{i} - 2\vec{j} + 6\vec{k}$

b) $4\vec{v} - 2\vec{u}$ where $\vec{v} = 2\vec{i} + 6\vec{j} - \vec{k}$ and $\vec{u} = \vec{j} + 3\vec{k}$

$4\vec{v} - 2\vec{u} = 4(2\vec{i} + 6\vec{j} - \vec{k}) - 2(\vec{j} + 3k)$
$= 8\vec{i} + 24\vec{j} - 4\vec{k} - 2\vec{j} - 6\vec{k}$
$= 8\vec{i} + 22\vec{j} - 10\vec{k}.$

A **unit vector** is a vector whose length is 1. For instance, $\vec{i}, \vec{j}$, and $\vec{k}$ are all unit vectors. In general, if $\vec{a} \neq \vec{0}$, then the unit vector that has the same direction as $\vec{a}$ is $\vec{u} = \frac{1}{|\vec{a}|}\vec{a} = \frac{\vec{a}}{|\vec{a}|}$.

Example 8: Find a unit vector $\vec{v}$ in the direction of the vector from $P(1,0,1)$ to $Q(3,2,0)$.

$\overrightarrow{PQ} = \langle 2,2,-1\rangle$

$|\overrightarrow{PQ}| = \sqrt{4+4+1} = 3$

unit vector $\vec{v} = \frac{1}{3}\langle 2,2,-1\rangle = \langle \frac{2}{3}, \frac{2}{3}, \frac{-1}{3}\rangle$

both ok

Historical note: The first mathematician to present a detailed theory of spaces of dimension great er than three was Hermann Grassmann (1809-1877). Grassmann was born and liven for most of his life in Stettin in Pomerania, now Szczecin, Poland. Although at the University of Berlin he mostly studied philology and theology, after leaving the university he returned to Stettin to pursue work in mathematics and physics to prepare himself to pass the state examination for teachers in those subjects. He subsequently taught briefly at a Berlin technical school and, after 1836, at various schools in his hometown. His great ambition in life was to qualify for a university position, but although he developed the ideas of the theory of **vector spaces**, few people read his efforts or recognized his great originality. Grassmann sent copies of his book to several influential mathematicians, but the only one who commented favorably on it was Hermann Hankel, a student of Riemann's, who planned to include some of Grassmann's material in his own book on complex variables. In the 1860's Grassmann turned his attention to the subject of languages and made some important scholarly contributions to the study of Sanskrit. His later mathematics works, however, were of lesser quality and he never attained his goal of a university professorship.[1]

[1] From A History of Mathematics, 3rd Ed. By Victor Katz. Pages 862-3.

Example 9: If $\vec{v} = 3\vec{i} - 4\vec{j}$ is a velocity vector, express it as a product of its speed times a unit vector in the direction of motion.

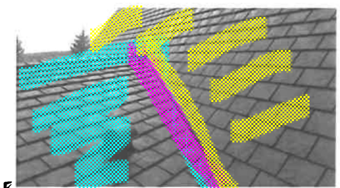
speed: $v = |\vec{v}| = \sqrt{3^2 + (-4)^2} = 5$

unit vector: $\frac{1}{5}(3\vec{i} - 4\vec{j}) = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$

Answer: $\vec{v} = 5\left(\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}\right)$

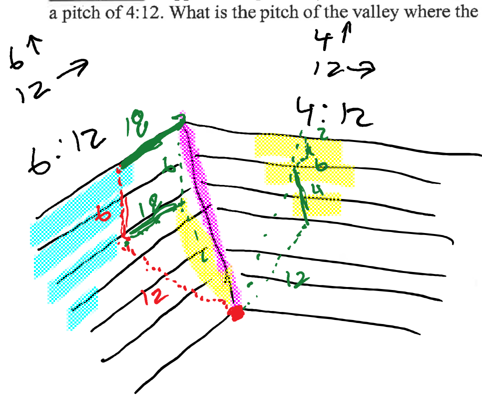
❖ **Applications**

Here is a preliminary example of an application that I learned about the hard way. A roof valley is where two roof planes meet.



Further, the pitch of a roof (steepness) in the U.S. is generally given as a ratio such as 4:12 which means that the roof has 4 inches of rise for every 12 inches of run. The second number is always given as 12.

Example 10: Suppose one plane of a roof has a 6:12 pitch and it is intersected by a second plane that has a pitch of 4:12. What is the pitch of the valley where the two planes meet?



$\langle 12, 0, 6\rangle + \langle 0, 18, 0\rangle$

valley along vector $\langle 12, 18, 6\rangle$

horizontal dist = 12

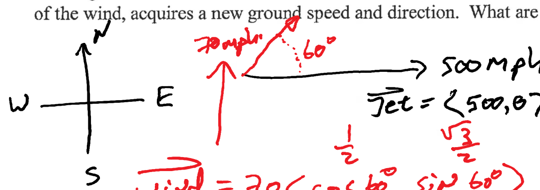
$\sqrt{12^2 + 18^2} = \sqrt{468}$

$\frac{12}{\sqrt{468}}\langle 12, 18, 6\rangle$

pitch $\frac{6 \cdot 12}{\sqrt{468}} : 12$

$3.33 : 12$ pitch

Example 10: A jet airliner, flying due east at 500 mph in still air, encounters a 70-mph tailwind blowing in the direction 60° north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?



$\vec{jet} = \langle 500, 0 \rangle$

$\vec{wind} = 70\langle \cos 60°, \sin 60° \rangle = \langle 35, 35\sqrt{3} \rangle$

$\vec{jet} + \vec{wind} = \langle 535, 35\sqrt{3} \rangle$

which has magnitude $\sqrt{535^2 + (35\sqrt{3})^2} \approx 538.4$

$\Rightarrow$ ground speed $= 538.4$ mph

$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{35\sqrt{3}}{535}\right) \approx 6.5°$ heading north of east.

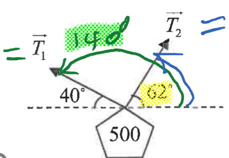
A very important application of vectors is **force**. This is a major theme in physics and engineering and will be revisited extensively in Calculus IV.

Force is also represented by a vector measured in Newtons (in the English system, pounds are a measure of force). If several forces are acting on an object, the result force experienced by the object is the vector sum of these forces.

Example 11: Lisa and Rose exercise at CrossFit. One of their trainings is to pick up a 500-lb weight (together) for a few seconds. Lisa is shorter than Rose, so they feel different amount of tension on their arms. Use the figure to find each tension.

Lisa: $|\vec{T}_1|\langle \cos 140°, \sin 140° \rangle = \vec{T}_1$

Rose: $\vec{T}_2 = |\vec{T}_2|\langle \cos 62°, \sin 62° \rangle$



$\langle 0, -500 \rangle$

At equilibrium the sum of the forces is $\vec{0} = \langle 0, 0 \rangle$.

x's: $|\vec{T}_1|\cos 140° + |\vec{T}_2|\cos 62° + 0 = 0$

y's: $|\vec{T}_1|\sin 140° + |\vec{T}_2|\sin 62° + (-500) = 0$

solve using a matrix.

create a coefficient matrix

$$\begin{bmatrix} \cos 140° & \cos 62° & 0 \\ \sin 140° & \sin 62° & 500 \end{bmatrix}$$

$$\text{RREF} \Rightarrow \begin{bmatrix} 1 & 0 & 240 \\ 0 & 1 & 392 \end{bmatrix}$$

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$|\vec{T}_1| = 240$ lbs, $|\vec{T}_2| = 392$ lbs.

Lisa Rose

In the last couple of examples, we have had to solve systems of equations. An efficient way to do this using a calculator involves matrices.

- **To Create a Matrix**

  Press MATRIX, EDIT, ENTER and put in the correct dimensions and fill in the elements

- **To Add, Subtract, Divide, or Multiply two matrices** ← matrix inverse.

  Press MATRIX, arrow down to the matrix you want, press ENTER
  This will put the matrix in your calculation screen
  Press the appropriate operation (+, -, /, *)
  Press MATRIX and arrow down to the matrix you want to add, press ENTER
  Press ENTER

- **To Solve Linear Equations using matrices**

  Create a coefficient matrix corresponding to the equation
  Press MATRIX, MATH, and arrow down to "rref" and press ENTER
  Press MATRIX, arrow down to the matrix you want and press ENTER
  Press ENTER

If you have a system:

$$\begin{cases} x + y = 1800 \\ 20x + 40y = 42000 \end{cases}$$

the "augmented matrix" is

$$\begin{bmatrix} 1 & 1 & 1800 \\ 20 & 40 & 42000 \end{bmatrix}$$

The "reduce row Echelon form" is

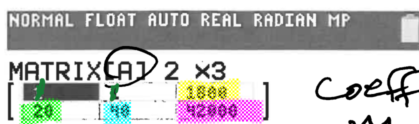
$$\begin{bmatrix} 1 & 0 & 1500 \\ 0 & 1 & 300 \end{bmatrix}$$

So the solution is:

$x = 1500$

$y = 300$

Creating an (augmented) matrix



coefficient Matrix

Reduced row echelon form (the solution)