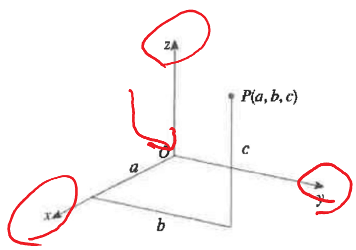
12.1: 3D Coordinate Systems

Friday, September 23, 2022 3:59 PM

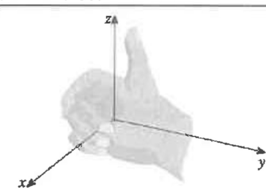
3D Coordinate Systems

As of now, all of our graphs in rectangular coordinate system were two-dimensional. We were locating a point on the *xy*-plane by its ordered pair (a,b). But we live in a three-dimensional world (physically of course!). To locate a point in the space we use ordered triple (a,b,c) and use 3 axes, *x*, *y* and *z*. These are three number lines that cross each other at their zero with $90°$ angles. This point is called the origin and has coordinates $(0,0,0)$.

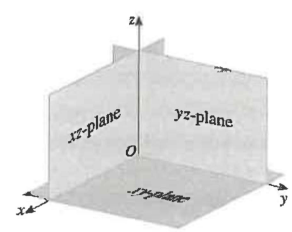


> **Historical note**: As we begin a discussion of geometric ideas, it may be helpful to learn that while you have previously learned about geometry and algebra separately, the approach we take in calculus is called analytic geometry and includes both geometry and algebra. Your skills in both areas will be useful.
>
> Analytic geometry was born in 1637 of two fathers, Rene' Descartes and Pierre de Fermat. Both Fermat and Descartes present the same basic techniques of relating algebra and geometry, the techniques whose further development culminated in the modern subject of analytic geometry. Both men came to the development of these techniques as part of the effort of rediscovering the "lost" Greek techniques of analysis. Both were intimately familiar with the Greek classics and in particular with Pappus. But Fermat and Descartes developed distinctly different approaches to their common subject, differences rooted in their differing points of view toward mathematics.[1]

The way we like to think of their position is shown in the diagram using the right-hand rule.

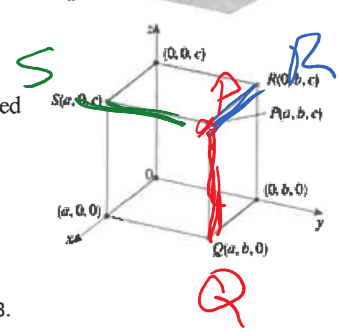


The *xy*-plane is the plane that contains the *x*- and *y*-axes; the *yz*-plane is the plane that contains the *y*- and *z*-axes; and the *xz*-plane is the plane that contains the *x*- and *z*-axes. These three coordinate planes divide space into eight parts, called **octants**. The **first octant,** in the foreground, is determined by the positive axes.



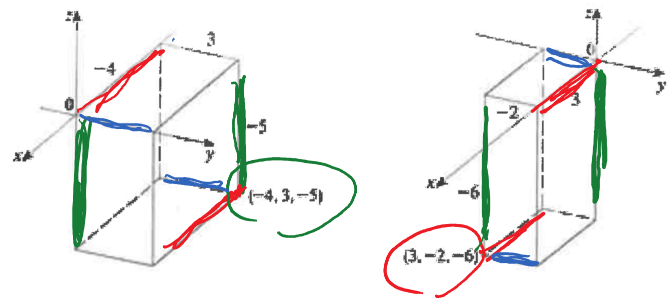
manipulate
13.26

The point $P(a,b,c)$ determines a rectangular box. If we drop a perpendicular from *P* to the *xy*-plane, we get a point $Q(a,b,0)$ called the **projection** of *P* onto the *xy*-plane, similarly $R(0,b,c)$ and $S(a,0,c)$ are projections onto the yz plane and xz plane, respectively.



[1] Abridged from A History of Mathematics, 3rd Ed. By Victor Katz. Page 473.

In the picture below, the points $(-4,3,-5)$ and $(3,-2,-6)$ are plotted.



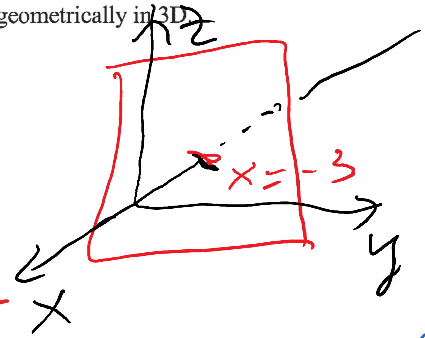
manipulate
13.27

Example 1: Interpret these equations and inequalities geometrically in 3D.

a) $x=-3$

no y,z

plane
thru
x = -3
parallel to
yz-plane

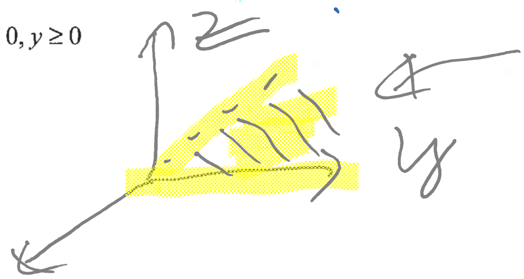


manipulate
13.30

b) $z \geq 0$

At or above
xy plane

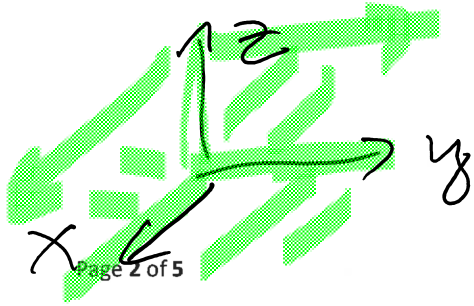
c) $z=0, x \leq 0, y \geq 0$



part of
xy plane

d) $x \geq 0, y \geq 0, z \geq 0$

first
octant



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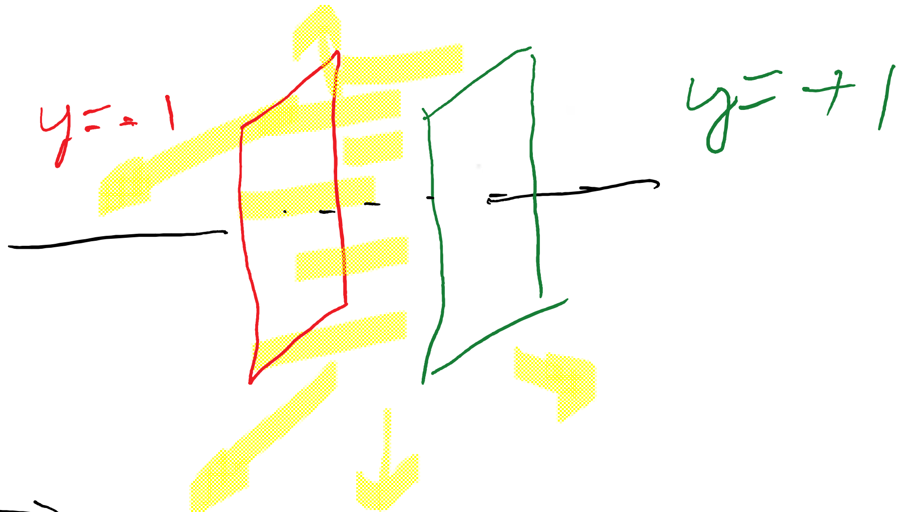
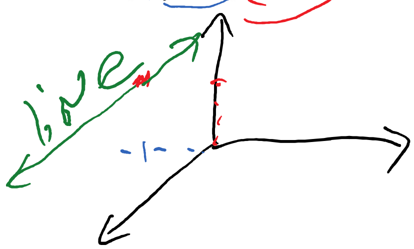
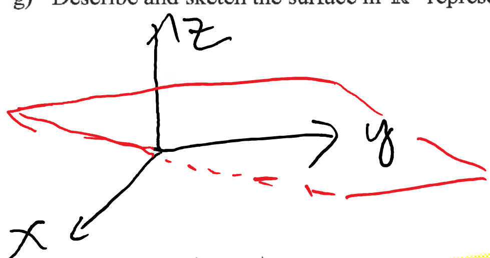
e) $-1 \le y \le 1$



f) $y=-2, z=2$



g) Describe and sketch the surface in $\mathbb{R}^3$ represented by the equation $x = z$



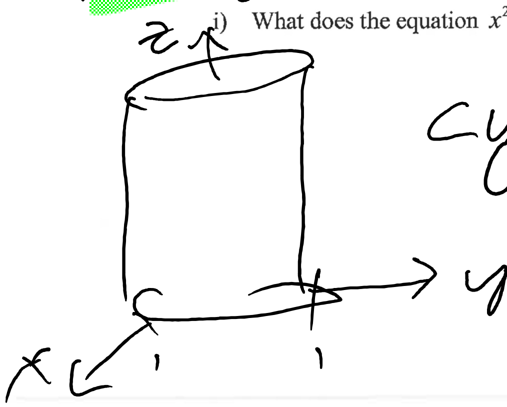
plane. Half way between xy-plane and yz-plane.

h) Which points (x, y, z) satisfy the equations $x^2 + y^2 = 4$ and $z = 5$

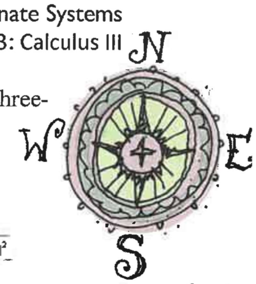
circle/cylinder

plane parallel to xy-plane

i) What does the equation $x^2 + y^2 = 1$ represent as a surface in $\mathbb{R}^3$



cylinder



Manipulate
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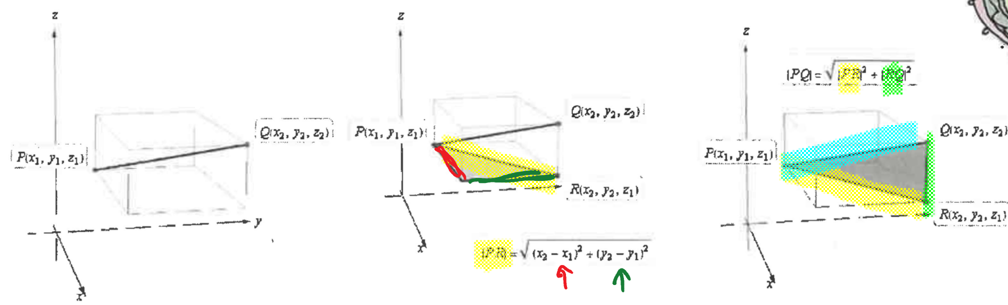
To create a formula for the **distance between two points** $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in three-dimensions, assume the given sketch (left).



$$|PQ| = \sqrt{|PR|^2 + |RQ|^2}$$
$$= \sqrt{\underbrace{(x_2 - x_1)^2 + (y_2 - y_1)^2}_{|PR|^2} + \underbrace{(z_2 - z_1)^2}_{|RQ|^2}}$$

Distance Formula in Three Dimensions: The distance $|PQ|$ between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example 2: Find the distance between $P_1(2, 1, 5)$ and $P_2(-2, 3, 0)$

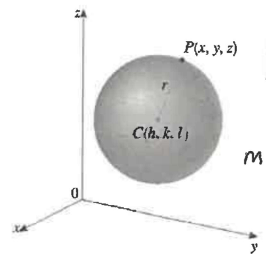
$$|P_1P_2| = \sqrt{(-2-2)^2 + (3-1)^2 + (0-5)^2}$$
$$= \sqrt{16 + 4 + 25}$$
$$= \sqrt{45}$$

A sphere of radius r and centered at $C(h,k,l)$ is the set of all points $P(x,y,z)$ whose distance from C is r. To find an equation for a sphere we focus on the fact that $|PC| = r$ so $|PC|^2 = r^2$ and hence:

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

Distance squared

always r^2.



manipulate 13, 34

> **Equation of a Sphere** An equation of a sphere with center $C(h, k, l)$ and radius r is
>
> $$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$
>
> In particular, if the center is the origin O, then an equation of the sphere is
>
> $$x^2 + y^2 + z^2 = r^2$$

Ex3: Find the center and radius of the sphere: $x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$

$$x^2 + 3x + \frac{9}{4} + y^2 + z^2 - 4z + 4 = +\frac{9}{4} + 4 - 1$$

$$\Rightarrow \left(x + \frac{3}{2}\right)^2 + y^2 + (z-2)^2 = \frac{21}{4}$$

$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$

$\left(\frac{4}{2}\right)^2 = 4$

sphere centered @ $\langle -\frac{3}{2}, 0, 2 \rangle$ w/ radius $\sqrt{21}/2$.

Ex4: Write an inequality to describe the solid upper hemisphere of the sphere of radius 4 centered at the origin.

$$x^2 + y^2 + z^2 \le 16 \quad \text{AND} \quad z \ge 0$$

solid

upper

$$\boxed{z \le +\sqrt{16 - x^2 - y^2}}$$