PS 3

Tuesday, November 8, 2022 6:38 PM

Power Series, part 3

❖ **Taylor Series and Approximations**

We are learning about power series. We have done this in two steps:

- We learned to find power series.
- We found where those series were convergent.

This second point and the interval of convergence provides us a clue toward our next step.

<u>Example 1</u>: Review two power series for $\frac{5}{4+x}$

$\frac{5}{4+x} = \sum_{n=0}^{\infty} 5(-1)^n (3+x)^n$ ← Interval of convergence: $-4 < x < -2$ with center at $x = -3$

and also

$\frac{5}{4+x} = \sum_{n=0}^{\infty} \frac{5}{4}\left(-\frac{x}{4}\right)^n$ ← Interval of convergence: $-4 < x < 4$ with center at $x = 0$

Notice that one function may have more than one power series representation. These series have different intervals of convergence. Of particular note, the center of the intervals of convergence are different.

> **<u>Historical Note</u>**: One of the most innovative aspects of Cauchy's [limit focused] program of rigor was his rejection of divergent series. These had been widely used in the eighteenth century, before Cauchy declared that they were unacceptably ill-defined, and produced ambiguous or even erroneous results. Picking up on this point in several papers of the early 1830s, Poisson tried to come to a clearer understanding of these series and the boundaries of their legitimacy. In his 1844 paper ''On Divergent Series and Various Points of Analysis Connected with Them,'' De Morgan blasted not only Poisson's ideas and Cauchy's definition of the integral on which they were based, but the whole preoccupation with certainty which valorized the search for rigor. ''Divergent series, at the time Poisson wrote, had been nearly universally adopted for more than a century, and it was only here and there that a difficulty occurred in using them,'' he fumed. The knowledgeable mathematician, De Morgan pointed out, could easily detect and correct such problems when they arose. To artificially control their use just in order to guarantee rigorous exactitude was at best unnecessary and ridiculous. At worst it could stand in the way of deeper understanding of the truth embodied in these series, which was as yet still poorly comprehended. As De Morgan wrote: "We must admit that many series are such as we cannot at present safely use, except as means of discovery, the results of which are to be subsequently verified. But to say that what we cannot use no others ever can, to refuse that faith in the future prospects of algebra which has already realized so brilliant a harvest . . . seems to me a departure from all rules of prudence." For De Morgan, to draw back from poorly defined or understood mathematical conclusions was a grievous error.[1]

[1] Joan Richards (2011) God, Truth and Mathematics in Nineteenth-century England, Theology and Science, 9:1, 53-74, DOI: 10.1080/14746700.2011.547005

This is not the first time we have generated a power series with a known center point. In particular, the Maclaurin Series formula $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ generates series centered at $x=0$. This formula (and the accompanying derivation) can be modified to generate series centered at $x=a$.

<u>Definition</u>: If *f* has a power series representation (expansion) at $x=a$, that is if $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ when $|x-a|<R$, then its coefficients are given by the formula $c_n=\frac{f^{(n)}(a)}{n!}$. We call the series above a Taylor Series.

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n \leftarrow \text{Maclaurin Series}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n \leftarrow \text{Taylor Series}$$

special case of the Taylor series when $a=0$.

<u>Example 2</u>: Find the Taylor Series for $f(x)=\frac{1}{\sqrt{x}}$ centered at $a=16$

$f(x)=\frac{1}{\sqrt{x}}=x^{-\frac{1}{2}}$ | At $a=16$: $\frac{1}{4}=\frac{1}{4}=\frac{1}{2^2}$

$f'(x)=-\frac{1}{2}x^{-3/2}$ | $-\frac{1}{2}\cdot\frac{1}{4^3}=-\frac{1}{128}=-\frac{1}{2^7}$

$f''(x)=\frac{1\cdot3}{2^2}x^{-5/2}$ | $\frac{1\cdot3}{2^2}\cdot\frac{1}{4^5}=\frac{1\cdot3}{4096}=\frac{1\cdot3}{2^{12}}$

$f'''(x)=-\frac{1\cdot3\cdot5}{2^3}x^{-7/2}$ | $-\frac{1\cdot3\cdot5}{2^3}\cdot\frac{1}{4^7}=\frac{-1\cdot3\cdot5}{131072}=\frac{-1\cdot3\cdot5}{2^{17}}$

$f^{(4)}(x)=\frac{1\cdot3\cdot5\cdot7}{2^4}x^{-9/2}$ | $\frac{1\cdot3\cdot5\cdot7}{2^4}\cdot\frac{1}{4^9}=\frac{1\cdot3\cdot5\cdot7}{4194304}=\frac{1\cdot3\cdot5\cdot7}{2^{22}}$

$$\frac{1}{\sqrt{x}}=\frac{1}{4}-\frac{\frac{1}{2}\cdot\frac{1}{4^3}}{1!}(x-16)+\frac{\frac{1\cdot3}{2^2}\cdot\frac{1}{4^5}}{2!}(x-16)^2-\frac{\frac{1\cdot3\cdot5}{2^3}\cdot\frac{1}{4^7}}{3!}(x-16)^3+\frac{\frac{1\cdot3\cdot5\cdot7}{2^4}\cdot\frac{1}{4^9}}{4!}(x-16)^4+\dots$$

Example 3: Find the Taylor Series for $g(x) = x - x^3$ centered at $a = 2$

	@ $a=2$
$g(x) = x - x^3$	-6
$g'(x) = 1 - 3x^2$	-11
$g''(x) = -6x$	-12
$g^{(3)}(x) = -6$	-6
$g^{(4)}(x) = 0$	0
$\vdots$	$\vdots$

$$g(x) = x - x^3$$

$$= \frac{-6}{0!} \frac{-11}{1!}(x-2) \frac{-12}{2!}(x-2)^2 - \frac{6}{3!}(x-2)^3 + 0 \ldots$$

Example 2 revisited: Answer the following for $f(x) = \frac{1}{\sqrt{x}}$

a.) Find the Taylor polynomials T_0, T_1, T_2, T_3 centered at $x = 16$.

T_N

$T_0 = \frac{1}{2^2}$

$T_1 = \frac{1}{2^2} - \frac{\frac{1}{2^7}}{1!}(x-16)$

$T_2 = \frac{1}{2^2} - \frac{\frac{1}{2^7}}{1!}(x-16) + \frac{\frac{1\cdot 3}{2^{12}}}{2!}(x-16)^2$

$T_3 = \frac{1}{2^2} - \frac{1}{2^7 \cdot 1!}(x-16) + \frac{1\cdot 3}{2^{12}\cdot 2!}(x-16)^2 - \frac{1\cdot 3\cdot 5}{2^{17}\cdot 3!}(x-16)^3$

Interesting math side note on how to write the product of evens or odds. Here are a couple of little examples:

Evens: $\underset{2\cdot 6}{12} \times \underset{2\cdot 5}{10} \times \underset{2\cdot 4}{8} \times \underset{2\cdot 3}{6} \times \underset{2\cdot 2}{4} \times \underset{2\cdot 1}{2} = 2^6 6!$

the odds are a little more difficult

Odds:

$$9 \times 7 \times 5 \times 3 \times 1 = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{8 \times 6 \times 4 \times 2} = \frac{9!}{2^4 4!}$$

$1\cdot 3\cdot 5\cdot 7\cdot 9 \cdots$

b.) Use T_0, T_1, T_2, T_3 to approximate $\frac{1}{\sqrt{16.8}}$ and $\frac{1}{\sqrt{22}}$. Then compare these approximations to the calculator value and list the errors. Note that absolute error is the difference between function value and the value that the Taylor Polynomial provides us.

	$\frac{1}{\sqrt{16.8}}$	$\frac{1}{\sqrt{22}}$	$\frac{1}{\sqrt{16.8}}$ error	$\frac{1}{\sqrt{22}}$ error
T_0	0.25	0.25	0.006	0.0368
T_1	0.2438	0.2031	0.0002	0.0101
T_2	0.2440	0.2163	0.00002	0.0031
T_3	0.2440	0.2122	0.00002	0.0010
	0.2440	0.2132		

error = |Actual − Approx|

small error

larger error

Taylor series centered @ $x = 16$

$\frac{1}{\sqrt{16.8}} \rightarrow x = 16.8$

$\frac{1}{\sqrt{22}} \rightarrow x = 22$

There is a problem. Taylor Polynomials (and Power Series) allow us to generate approximate values. However, these are *estimates*. How close are the approximate values to the actual values? In order to find the error, we would need to know the exact value but if we knew the exact values we would not need an estimate in the first place.

We need a way to find the error that does not require that we know the exact value. Or, to be more precise, we need a way to bound the error.

This requires that we build up more notation.

Definition: Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing a. For all x in I, $f(x) = T_n(x) + R_n(x)$ where T_n is the nth-degree Taylor Polynomial for f centered at $x = a$ and $R_n(x)$ is the remainder.

To be clear: $\underbrace{f(x)}_{\text{exact value}} = \underbrace{T_n(x)}_{\text{approximate value}} + \underbrace{R_n(x)}_{\text{remainder or error}}$

To be clear: $\underbrace{f(x)}_{\text{exact value}} = \underbrace{T_n(x)}_{\text{approximate value}} + \underbrace{R_n(x)}_{\text{remainder or error}}$

Example 2 revisited: Find $R_3(x)$ for the 3rd degree Taylor Polynomial of $f(x)=\frac{1}{\sqrt{x}}$ centered at $x=16$.

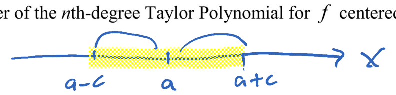
$$\frac{1}{\sqrt{x}} = \underbrace{\frac{1}{2^2} - \frac{1}{2^7\cdot 1!}(x-16) + \frac{1\cdot 3}{2^{12}\cdot 2!}(x-16)^2 - \frac{1\cdot 3\cdot 5}{2^{17}\cdot 3!}(x-16)^3}_{T_3} \underbrace{+ \frac{1\cdot 3\cdot 5\cdot 7}{2^{22}\cdot 4!}(x-16)^4 - \cdots}_{R_3} \rightarrow$$

$$R_3(x) = \frac{1\cdot 3\cdot 5\cdot 7}{2^{22}\cdot 4!}(x-16)^4 - \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{2^{27}\cdot 5!}(x-16)^5 + \frac{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11}{2^{32}\cdot 6!}(x-16)^6 - \cdots$$

remainder or error

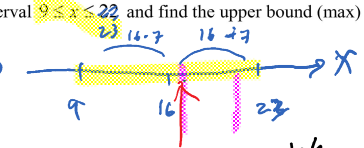
Notice: Finding $R_3(x)$ requires knowing the exact value of $\frac{1}{\sqrt{x}}$. In this case we can use our calculator to evaluate $\frac{1}{\sqrt{x}}$, but what if we didn't have that ability?

The Remainder Estimation Theorem: Suppose there exists a number M such that $\left|f^{(n+1)}(x)\right| \le M$ for all x in the interval $[a-c, a+c]$. The remainder of the nth-degree Taylor Polynomial for f centered at a satisfies: $\left|R_n(x)\right| \le M\dfrac{|x-a|^{n+1}}{(n+1)!}$



Example 2 revisited: Bound the error in T_3 on the interval $9 \le x \le 22$ and find the upper bound (max) error in your estimates for $\dfrac{1}{\sqrt{16.8}}$ and $\dfrac{1}{\sqrt{22}}$.

23

① 

② Find M.

$\left|f^{(3+1)}(x)\right| \le M$ on $[9, 23]$

Scratch

$f \quad x^{-1/2}$

$f' \quad -\frac{1}{2}x^{-3/2}$

$f'' \quad \frac{3}{4}x^{-5/2}$

$f''' \quad \frac{3\cdot 5}{8}x^{-7/2}$

$f^{(4)}(x) = -\frac{3\cdot 5\cdot 7}{16}x^{-9/2}$

$\Rightarrow \left|\dfrac{-3\cdot 5\cdot 7\,x^{-9/2}}{16}\right| \le M$

$\Rightarrow \left|\dfrac{105}{16x^{9/2}}\right| \le M$ on $[9, 23]$

greatest when x is smallest $(x=9)$

Let $M = \dfrac{105}{16(9)^{9/2}} \approx 0.0003334$

③ Bound the remainder/error.

$|R_3(x)| \le M\dfrac{|x-16|^4}{4!}$

evaluate when
$x = 16.8$
and
$x = 22$

① $|R_3(x)| \le 0.0003334\dfrac{|16.8-16|^4}{4!}$

≈ 0.00000569 (very small error).

② $|R_3(x)| \le 0.0003334\dfrac{|22-16|^4}{4!}$

$= 0.018$ (larger max error).

Example 3: Approximate $\int_0^1 e^{-x^2}\,dx$ to within 0.001

① recall: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$\Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

② $$\int e^{-x^2}dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}dx$$

$$= \sum_{n=0}^{\infty} \int \frac{(-1)^n x^{2n}}{n!}dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}$$

③ $$\int_0^1 e^{-x^2}dx = \left[C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\right]_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} \longleftarrow \text{exact}$$

④ Approx w/in 0.001. Alternating series estimation theorem.

$$\int_0^1 e^{-x^2}dx = \underbrace{+1 - \frac{1}{3} + \frac{1}{5\cdot 2!} - \frac{1}{7\cdot 3!} + \frac{1}{9\cdot 4!}}_{0.7475} - \frac{1}{11\cdot 5!} + \cdots$$

0.7475 ± 0.0008

Bonus Questions

Example 4: Find the first three non-zero terms in the Maclaurin series for $f(x) = e^x \sin x$.

① recall: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

② multiply. $e^x \sin x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots$

$- \frac{x^3}{3!} - \frac{x^4}{3!} - \frac{x^5}{3!} + \ldots$

$+ \frac{x^5}{5!} + \frac{x^6}{5!} + \frac{x^7}{5!} + \ldots$

③ Add $e^x \sin x \approx x + x^2 + \frac{1}{3}x^3 + 0x^4 - \frac{19}{120}x^5 + \ldots$

Example 5: Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

Ratio test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)^{n+1}}}{\frac{(x-2)^n}{n^n}}\right|$

$= \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^{n+1}}\cdot\frac{n^n}{(x-2)^n}\right|$

$= \lim_{n\to\infty}\left|\frac{(x-2)\cdot n^n}{(n+1)^{n+1}}\right|$

$= |x-2| \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^{n+1}}\right| \leftarrow (n+1)^{n+1} = (n+1)^n(n+1)$

$= |x-2| \lim_{n\to\infty}\frac{n^n}{(n+1)^n(n+1)}$ $\to \infty$, $\to \infty$

$= |x-2| \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}$

$= |x-2|\, e^{\lim_{n\to\infty}\ln\left(\left(\frac{n}{n+1}\right)^n\cdot\frac{1}{n+1}\right)}$

$= |x-2|\, e^{\lim_{n\to\infty}\left[\ln\left[\left(\frac{n}{n+1}\right)^n\right]+\ln\left(\frac{1}{n+1}\right)\right]}$

$= |x-2|\, e^{\lim_{n\to\infty} n\ln\left(\frac{n}{n+1}\right)} \cdot e^{\lim_{n\to\infty}\ln\left(\frac{1}{n+1}\right)}$

$= |x-2|\, e^{\lim_{n\to\infty}\frac{\ln\left(\frac{n}{n+1}\right)}{\frac{1}{n}}} \cdot \lim_{n\to\infty}\frac{1}{n+1} \to 0$

(H) $= |x-2|\, e^{\lim_{n\to\infty}\frac{\frac{n+1}{n}\cdot\frac{1(n+1)-1(n)}{(n+1)^2}}{-\frac{1}{n^2}}} \cdot 0$

$= |x-2|\, e^{\lim_{n\to\infty}\left(-\frac{n^3}{1}\cdot\cancel{\frac{n+1}{n}\cdot\frac{1}{(n+1)^2}}\right)}$ $\to -1$

$= |x-2|\, e^{-1}\cdot 0$

$= 0 < 1.$

The series converges $(-\infty,\infty)$

$R=\infty.$

Example 6: Find the Taylor series for $f(x)=\cos x$ centered at $x=\pi$. Graph f and T_4 together.

T_4 = 4th degree Taylor polynomial.

fcts	@ $x=\pi$
$f(x)=\cos x$	-1
$f' = -\sin x$	0
$f'' = -\cos x$	1
$f^{(3)} = \sin x$	0
$f^{(4)} = \cos x$	-1
$f^{(5)} = -\sin x$	0

$\cos x = -1 + 0(x-\pi) + \frac{1}{2!}(x-\pi)^2 + \frac{0}{3!}(x-\pi)^3 - \frac{1}{4!}(x-\pi)^4 + \ldots$

$T_4(x) = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4$

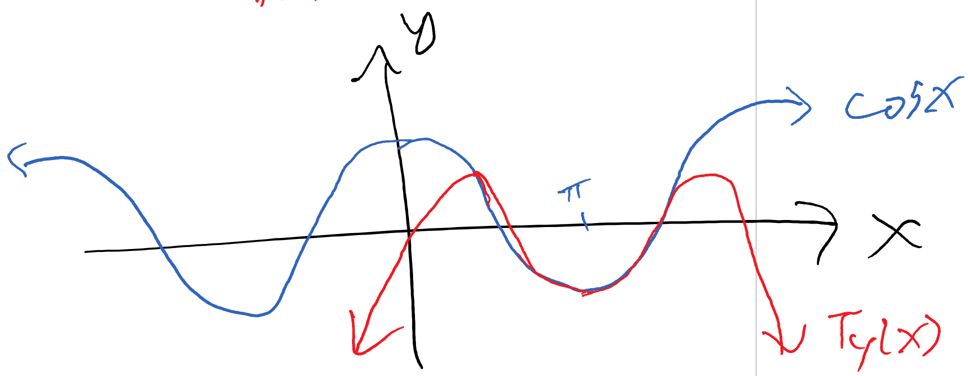
so

$\cos x = -1 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 - \frac{1}{8!}(x-\pi)^8 + \ldots$

$f(x)$

y

$f^{(5)} = -\sin x$ 1   0 $f(x)$



Question: Why would it be beneficial to center the series at $x = \pi$?

if you are interested in values of cos x near $x = \pi$.

ex: cos(3), cos(2.9), cos(3.14)

Example 7: Determine the number of terms of the Maclaurin Series for e^x that should be used to estimate $e^{0.1}$ to within 0.00001.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$e^{0.1} = 1 + 0.1 + \frac{0.1^2}{2} + \frac{0.1^3}{3!} + \frac{0.1^4}{4!} + \dots$$

① find $M \geq |\underbrace{f^{(n+1)}(x)}_{e^x}|$ on same interval.

$-\ln 2 < x < \ln 2$

$M \geq e^{\ln 2} = 2$

choose $M = 2$

② find $|R_n(x)| \leq \frac{M\,|x-0|^{n+1}}{(n+1)!}$ given $x = 0.1$

$$\Rightarrow |R_n(x)| \leq \frac{2\,(0.1)^{n+1}}{(n+1)!} \leq 0.00001$$

n	max error	
1	$\frac{2(0.1)^2}{2!} = 0.01$	bigger than 0.00001 (fail)
2	$\frac{2(0.1)^3}{3!} = 0.000\bar{3}$	bigger than 0.00001 (fail)
3	$\frac{2(0.1)^4}{4!} = 0.000008\bar{3}$	smaller than 0.00001 SUCCESS!

Need $n = 3$ terms.

$$e^{0.1} \approx \underbrace{1 + \frac{0.1}{1} + \frac{0.1^2}{2} + \frac{0.1^3}{6}}_{1.10516} \pm 0.00001$$

1.10516 ± 0.00001

calculator: $e^{.1} = 1.105170918$