PS 2

Wednesday, November 2, 2022 3:53 PM

Power Series, part 2

❖ **Power Series term by term using derivatives**

We have found the derivative of power series. This next example is a little different; here we use derivatives to find the coefficients.

Example 1: Find a power series representation for $f(x) = \sin(x)$.

Suppose $\sin x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

If True, then true for all x's

If $x = 0$: $\boxed{0 = c_0} + c_1(0) + c_2(0) + \dots$

Differentiate both sides

$\Rightarrow \cos x = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$

If $x = 0$: $\boxed{1 = c_1} + 0 + 0 + \dots$

Differentiate both sides WRT x

$\Rightarrow -\sin x = 2c_2 + 6c_3 x + 12 c_4 x^2 + \dots$

If $x = 0$. $0 = 2c_2 + 0 \Rightarrow \boxed{c_2 = 0}$

Differentiate ...

$\Rightarrow -\cos x = 6c_3 + 24 c_4 x + 60 c_5 x^2 + \dots$

If $x = 0$: $-1 = 6c_3 \Rightarrow c_3 = \frac{1}{6} = \frac{1}{3\,2\,1}$

$\sin 0$

$\cos 0$

$-\sin 0$

$-\cos 0$

To date

$c_0 = 0$, $c_1 = \frac{1}{1!}$, $c_2 = 0$; $c_3 = \frac{(-)1}{3!}$

$\sin x = \frac{1}{1!}x^1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$

or $\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$

The series we found is an example of a more general type of power series called a **Maclaurin Series**.

Definition: Suppose the function f has derivatives of all orders on an interval centered at $x = 0$, then its **Maclaurin Series** is:

zero

$$f(x) = f(0) + f'(o)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots$$

This can be written more concisely as: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$

$$n=0$$

The series we found is an example of a more general type of power series called a **Maclaurin Series**.

Definition: Suppose the function f has derivatives of all orders on an interval centered at $x=0$, then its **Maclaurin Series** is:

zero

$$f(x)=f(0)+f'(o)x+\frac{f''(0)}{2!}x^2+\frac{f^{(3)}(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\dots$$

This can be written more concisely as: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$

Note: A Maclaurin Series is a type of power series. It is found by finding the coefficients term by term using derivatives.

Example 2: Find a Maclaurin Series (that is, a power series) representation for $f(x)=e^x$.

$$c_0=\frac{f(0)}{0!}=\frac{1}{1}=1 \qquad c_3=\frac{f^{(3)}(0)}{3!}=\frac{1}{3!}$$

$$c_1=\frac{f'(0)}{1!}=\frac{1}{1}=1 \qquad a_4=\frac{f^{(4)}(0)}{4!}=\frac{1}{4!}$$

$$c_2=\frac{f''(0)}{2!}=\frac{1}{2!} \qquad c_n=\frac{1}{n!}$$

thus

$$e^x=1+1x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{x^4}{4!}+\frac{x^5}{5!}\dots$$

$$\boxed{e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}}$$

included in the soln to $f(x)=f'(x)$

As with geometric series, the Maclaurin Series can be manipulated to go quite a way:

Example 3: Find a Maclaurin Series (that is, a power series) representation for the following:

a.) $f(x)=e^{2x}$

recall $e^{x}=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$\Rightarrow e^{2x}=\sum_{n=0}^{\infty}\frac{(2x)^n}{n!}=\sum_{n=0}^{\infty}\frac{2^n x^n}{n!}$

b.) $g(x)=x^4e^x$

recall $e^{x}=\sum_{n=0}^{\infty}\frac{x^n}{n!}$ ✓

$\Rightarrow x^4e^x=x^4\sum_{n=0}^{\infty}\frac{x^n}{n!}$

$=\sum_{n=0}^{\infty}\frac{x^4x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{n+4}}{n!}$

Three Maclaurin Series to Memorize:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}\qquad \sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\qquad \cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

odd fct even fct

If you have been paying attention, you may have noticed that we have found power series in the previous few examples, but have NOT made mention of where these series are valid. This is because we are no longer working with geometric series and consequently need more power (pun).

Specifically, we will use two facts and one method/test.

Fact 1: The Harmonic Series diverges. In symbols: $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\dots=\infty$

not important

Fact 2: The *Alternating* Harmonic Series converges. In symbols: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\ln(2)$

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$ converges

So why does one infinite series converge and another diverge? That is a big question and we will leave it for another course. However, we can see that sometimes we sum an infinite number of terms and get a number (converge) and other times we don't get a number (diverge).

One of the most powerful ways of determining if a series will converge is to ask, "Do the terms decrease fast enough to converge? But how do we measure, "Fast enough"?

Example 4: Explore the ratio of consecutive terms on these three series

a.) $\sum_{n=0}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^n = 2 + 2\left(\frac{1}{3}\right) + 2\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)^3 + 2\left(\frac{1}{3}\right)^4 + \cdots$

geometric series

$\sum_{n=0}^{\infty} 2\left(\frac{1}{3}\right)^n = \frac{2}{1-1/3} = 3$

ratio: $\frac{2\left(\frac{1}{3}\right)^4}{2\left(\frac{1}{3}\right)^3} = \frac{1}{3}$

notice: $\frac{a_{n+1}}{a_n} = \frac{1}{3}$

b.) $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots$

$\frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$ $\qquad$ $\frac{\frac{1}{6}}{\frac{1}{5}} = \frac{5}{6}$ $\qquad$ $\frac{\frac{1}{n+1}}{\frac{1}{n}} = \frac{n}{n+1}$

As $n \to \infty$ $\frac{n}{n+1} = 1$

c.) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \cdots$

$\frac{\frac{x^3}{3!}}{\frac{x^2}{2!}} = \frac{x}{3}$ $\qquad$ $\frac{\frac{x^6}{6!}}{\frac{x^5}{5!}} = \frac{x}{6}$ $\qquad$ $\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x}{n+1}$

As $n \to \infty$ for a fixed value of x, we know $\frac{x}{n+1} \longrightarrow 0$