PS 3

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As with geometric series, the Maclaurin Series can be manipulated to go quite a way:

Example 3: Find a Maclaurin Series (that is, a power series) representation for the following:

a.) $f(x)=e^{2x}$

b.) $g(x)=x^4e^x$

Three Maclaurin Series to Memorize:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

❖ **A deep dive into where Power Series Converge**

If you have been paying attention, you may have noticed that we have found power series in the previous few examples, but have NOT made mention of where these series are valid. This is because we are no longer working with geometric series and consequently need more power (pun).

Specifically, we will use two facts and one method/test.

Fact 1: The Harmonic Series diverges. In symbols: $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots=\infty$$

Harmonic series diverges.

Fact 2: The *Alternating* Harmonic Series converges. In symbols: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=\ln(2)$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\dots$$

So why does one infinite series converge and another diverge? That is a big question and we will leave it for another course. However, we can see that sometimes we sum an infinite number of terms and get a number (converge) and other times we don't get a number (diverge).

One of the most powerful ways of determining if a series will converge is to ask, "Do the terms decrease fast enough to converge? But how do we measure, "Fast enough"?

Example 4: Explore the ratio of consecutive terms on these series

a.) $\sum_{n=0}^{\infty} 2\cdot\left(\frac{1}{3}\right)^n = 2\cdot\left(\frac{1}{3}\right)^0 + 2\cdot\left(\frac{1}{3}\right)^1 + 2\cdot\left(\frac{1}{3}\right)^2 + 2\cdot\left(\frac{1}{3}\right)^3 + 2\cdot\left(\frac{1}{3}\right)^4 + \ldots + 2\cdot\left(\frac{1}{3}\right)^n + 2\cdot\left(\frac{1}{3}\right)^{n+1} + \ldots$

Converges

$\frac{2(\frac{1}{3})^1}{2(\frac{1}{3})^0} = \frac{1}{3}$ $\quad \frac{2(\frac{1}{3})^4}{2(\frac{1}{3})^3} = \frac{1}{3}$ $\quad \frac{2(\frac{1}{3})^{N+1}}{2(\frac{1}{3})^N} = \frac{1}{3}$

and it appears that as $n \to \infty$, the ratio of consecutive terms Approaches $\frac{1}{3}$

b.) $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n} + \frac{1}{n+1} + \ldots$

Diverges

$\frac{\frac{1}{2}}{\frac{1}{1}} = \frac{1}{2}$ $\quad \frac{\frac{1}{5}}{\frac{1}{4}} = \frac{4}{5}$ $\quad \frac{\frac{1}{N+1}}{\frac{1}{N}} = \frac{N}{N+1} \longrightarrow 1$

and it appears that as $n \to \infty$, the ratio of consecutive terms approaches 1

c.) $\sum_{n=0}^{\infty} 2\cdot 3^n = 2\cdot 3^0 + 2\cdot 3^1 + 2\cdot 3^2 + 2\cdot 3^3 + 2\cdot 3^4 + \ldots + 2\cdot 3^n + 2\cdot 3^{n+1} + \ldots$

Diverges

$\frac{2\cdot 3^1}{2\cdot 3^0} = 3$ $\quad \frac{2\cdot 3^4}{2\cdot 3^3} = 3$ $\quad \frac{2\cdot 3^{N+1}}{2\cdot 3^N} = 3$

and it appears that as $n \to \infty$, the ratio of consecutive terms approaches 3

d.) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots + \frac{(-1)^{n+1}}{n} + \frac{(-1)^{n+2}}{n+1} + \ldots$

Converges

$\left|\frac{-\frac{1}{2}}{\frac{1}{1}}\right| = \left|-\frac{1}{2}\right|$ $\quad \left|\frac{\frac{1}{5}}{-\frac{1}{4}}\right| = \left|-\frac{4}{5}\right|$ $\quad \left|\frac{-\frac{1}{N+1}}{\frac{1}{N}}\right| = \left|-\frac{N}{N+1}\right| \longrightarrow 1$

and it appears that as $n \to \infty$, the ratio of consecutive terms approaches 1

e.) $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \ldots$

converges to e^x.

$\left|\frac{x}{1}\right|$ $\quad \left|\frac{\frac{x^4}{4!}}{\frac{x^3}{3!}}\right| = \left|\frac{3!\,x}{4!}\right| = \left|\frac{x}{4}\right|$ $\quad \left|\frac{\frac{x^{N+1}}{(N+1)!}}{\frac{x^N}{N!}}\right| = \left|\frac{x}{N+1}\right| = \frac{1}{N+1}|x| \to 0$

and it appears that as $n \to \infty$, the ratio of consecutive terms approaches 0

?

Let's sum up what we have seen about series and ratios thus far.

1. Power series, coefficients, and terms
   a. A power series is of the form: $\sum_{n=0}^{\infty} c_n x^n = \underbrace{c_0}_{a_0} + \underbrace{c_1 x}_{a_1} + \underbrace{c_2 x^2}_{a_2} + \underbrace{c_3 x^3}_{a_3} + \ldots + \underbrace{c_n x^n}_{a_n} + \underbrace{c_{n+1} x^{n+1}}_{a_{n+1}} + \ldots$

      No x's

      include x's

   b. The coefficients are: $c_0, c_1, c_2, c_3, \ldots, c_n, c_{n+1}, \ldots$
   c. The terms are: $a_0, a_1, a_2, a_3, \ldots, a_n, a_{n+1}, \ldots$
2. About ratios of consecutive terms a_n and a_{n+1}:
   a. Ratios > 1 mean the terms are increasing quickly.
   b. Ratios < 1 mean the terms are decreasing to zero rapidly.
   c. Ratios ≈ 1 aren't changing quickly enough to know to draw a conclusion (using ratios).
3. About absolute value
   a. Terms can vary in sign. The absolute value of the ratio sometimes makes quantities bigger (positive) and thus convergence more difficult. So if a series converges with the absolute value, it certainly converges without it.

This leads us to a powerful and (relatively) easy test for convergence.

Definition: The Ratio Test

a.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n x^n$ is (absolutely) convergent.

b.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n x^n$ divergent.

c.) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, then the Ratio Test is inconclusive. Try again.

Review the questions on the previous page. Notice how they provide examples of the Ratio Test at work. In particular, notice that when the ratio approaches 1 this can mean either convergence or divergence. That is why we call the test "inconclusive" in this instance.

Example 4: Where does the Maclaurin series $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Ratio Test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|$

$= \lim_{n\to\infty}\left|\frac{x}{n+1}\right|$

$= 0 \leftarrow$ for all x.

The series converges for values of x. $(-\infty, \infty)$.

Example 5: Consider $\dfrac{5}{4-x}=\dfrac{5}{1-(-3-x)}$ with geometric series $\sum_{n=0}^{\infty}5(-3-x)^n=\sum_{n=0}^{\infty}5(-1)^n(3+x)^n$.

Find the x values where this series converges using the ratio test.

ratio test: $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\dfrac{\cancel{5}(-1)^{n+1}(3+x)^{n+1}}{\cancel{5}(-1)^n(3+x)^n}\right|$

$=\lim_{n\to\infty}|3+x|$

series converges when $|3+x|<1$

$\Rightarrow -1<3+x<1$

$\Rightarrow -4<x<-2$

Test $x=-2$ Diverge

Test $x=-4$ Diverge

As we focus in on convergence, two definitions will help us.

Definition: The interval of convergence of a power series is the interval that consists of all values of x for which a series converges.

Note: Intervals have **endpoints** that may (or may not) be included in the interval of convergence.

Lazy Definition: The radius of convergence is half the width of the interval of convergence (possibly zero or infinity).

Example 5 revisited: Consider $\dfrac{5}{4-x}=\sum_{n=0}^{\infty}5(-1)^n(3+x)^n$ and find its interval of convergence and radius of convergence.

width = 2

I.O.C. $-4<x<-2$ or $(-4,-2)$

R.O.C. $R=\dfrac{2}{2}=1$

The following examples show how we find the interval of convergence and radius of convergence once we have a power series in hand.

Example 6: Suppose you are given a power series $\sum_{n=1}^{\infty}\frac{x^n}{n3^n}$. Find its interval of convergence and radius of convergence.

ratio test: $\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)3^{n+1}}}{\frac{x^n}{n3^n}}\right| = \lim_{n\to\infty}\left|\frac{x\cdot n}{3(n+1)}\right| = \left|\frac{x}{3}\right|$

solve $\left|\frac{x}{3}\right|<1$

$\Rightarrow |x|<3$

$\Rightarrow -3<x<3$

Test $x=-3$: $\sum_{n=1}^{\infty}\frac{(-3)^n}{n3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ converges

Test $x=3$: $\sum_{n=1}^{\infty}\frac{3^n}{n\cdot 3^n} = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges

conclusion
I.O.C.: $[-3,3)$
R.O.C.: $R=3$

Example 7: Suppose you are given a power series $\sum_{n=0}^{\infty}\frac{(-3)^n x^{2n}}{\sqrt{n+1}}$. Find its ~~interval of convergence~~ and radius of convergence.

ratio test: $\lim_{n\to\infty}\left|\frac{\frac{(-3)^{n+1}x^{2(n+1)}}{\sqrt{n+2}}}{\frac{(-3)^n x^{2n}}{\sqrt{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{2n+2}}{\sqrt{n+2}}\cdot\frac{\sqrt{n+1}}{(-3)^n x^{2n}}\right|$, endpoints are beyond us

$=\lim_{n\to\infty}\left|\frac{3x^2\sqrt{n+1}}{\sqrt{n+2}}\right|$

$=|3x^2|<1$

$\Rightarrow |x^2|<\frac{1}{3}$

$\Rightarrow |x|<\frac{1}{\sqrt{3}}$

$\Rightarrow -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$

R.O.C. $R=\frac{1}{\sqrt{3}}$

Example 8: Find the interval of convergence and radius of convergence of the Maclaurin series

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

ratio test: $\lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{(-1)^n x^{2n+1}}{(2n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\cdot\frac{(2n+1)!}{(-1)^n x^{2n+1}}\right|$

$=\lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+1)}\right|$

$=0<1$

I.O.C. $(-\infty,\infty)$
R.O.C. $R=\infty$.