PS 1

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Power Series, part 1

❖ **Power Series by the pictures**

Intuitively, a **power series** is like an infinitely long polynomial (except that polynomials are defined so as to have finite length). Examples include:

a.) $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \dots + (n+1)x^n + \dots = \sum_{n=0}^{\infty} (n+1)x^n$

$n=0$

b.) $g(x) = x - x^3 + x^5 - x^7 + x^9 - x^{11} + x^{13} - x^{15} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} x^{2n-1}$

$g(x) = 0 + x + 0x^2 - x^3 + 0x^4 + x^5 + 0x^6 - x^7 + \dots$

c.) $h(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 + \frac{1}{6}x^5 + \dots = \sum_{n=0}^{\infty} \frac{1}{n+1} x^n$

c_4 c_5 c_n

Key idea: Working with power series $c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n + \dots$ is first and foremost about finding the coefficients $c_0, c_1, c_2, c_3, \dots, c_n, \dots$ Often (at least in class) we find a nice formula for the coefficients and simply write $\sum_{n=0}^{\infty} c_n x^n$. But, either way the task at hand begins with finding coefficients.

About now, you might be wondering why anyone would care about power series. The next example provides a graphical connection between power series and more familiar topics.

Example 1: Use a graph to explore $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$$\sum_{n=0}^{1} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{1 \cdot 1}{1} + \frac{(-1)x^2}{2}$$

$n=0$

$$\sum_{n=0}^{2} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{1 \cdot x^4}{24}$$

$$\sum_{n=0}^{3} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}$$

$0! = 1$
$1! = 1$
$2! = 2 \cdot 1$
$3! = 3 \cdot 2 \cdot 1$
$4! = 4 \cdot 3 \cdot 2 \cdot 1$
$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
$\vdots$

Polynomial functions can be evaluated using basic operations (addition, subtraction, multiplication and division) and they can be differentiated / integrated pretty easily. But this is not the same for many other functions such as trig, exponential or logarithmic functions! It is beneficial to rewrite a function as a polynomial. This strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions values. Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.

❖ **The Geometric Series**

There are infinitely many power series, but some are famous enough to merit a name. The first of these is named the **geometric series**. The next few examples help us understand this very important (but basic) example.

We will begin with examples without *x* and then work our way toward actual **power series**

<u>Example 2</u>: Evaluate the following:

a.) $\sum_{n=0}^{5} 2\cdot 3^n = 2\cdot 3^0 + 2\cdot 3^1 + 2\cdot 3^2 + 2\cdot 3^3 + 2\cdot 3^4 + 2\cdot 3^5$

$= 728$

b.) $\sum_{n=0}^{5} 2\cdot\left(\frac{1}{3}\right)^n = 2\left(\frac{1}{3}\right)^0 + 2\left(\frac{1}{3}\right)^1 + 2\left(\frac{1}{3}\right)^2 + \dots + 2\left(\frac{1}{3}\right)^5$

$= \frac{728}{243} \approx 2.999$

Example 3: The upper limit in the sum is important. Compare $\sum_{n=0}^{50} 2\cdot 3^n$ and $\sum_{n=0}^{50} 2\cdot\left(\frac{1}{3}\right)^n$

$$\sum_{n=0}^{50} 2\cdot 3^n = 2\cdot 3^0 + 2\cdot 3^1 + \ldots + 2\cdot 3^{50}$$

big like 3 ⇒ crazy

$\approx 1.44\times 10^{24}$

$$\sum_{n=0}^{50} 2\left(\tfrac{1}{3}\right)^n = 2\left(\tfrac{1}{3}\right)^0 + 2\left(\tfrac{1}{3}\right) + \ldots + 2\left(\tfrac{1}{3}\right)^{50}$$

small like $\left(\frac{1}{3}\right)$ ⇒ nice

$\approx 7.79\times 10^{-24}$

≈ 0

And most interesting is when we allow the upper limit to be infinite in which case we are left with what is called an **infinite series**.

Example 4: Evaluate the infinite series $\sum_{n=0}^{\infty} 2\cdot\left(\frac{1}{3}\right)^n$

$$\sum_{n=0}^{\infty} 2\left(\tfrac{1}{3}\right)^n = \lim_{k\to\infty}\sum_{n=0}^{k} 2\left(\tfrac{1}{3}\right)^n$$

$$= \lim_{k\to\infty}\left[2\left(\tfrac{1}{3}\right)^0 + 2\left(\tfrac{1}{3}\right)^1 + 2\left(\tfrac{1}{3}\right)^2 + \ldots + 2\left(\tfrac{1}{3}\right)^k\right] 1$$

$$= \lim_{k\to\infty} 2\left[\left(\tfrac{1}{3}\right)^0 + \left(\tfrac{1}{3}\right)^1 + \left(\tfrac{1}{3}\right)^2 + \ldots + \left(\tfrac{1}{3}\right)^k\right]\frac{1-\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \lim_{k\to\infty} 2\,\frac{\left[\left(\tfrac{1}{3}\right)^0 + \cancel{\left(\tfrac{1}{3}\right)^1} + \cancel{\left(\tfrac{1}{3}\right)^2} + \ldots + \cancel{\left(\tfrac{1}{3}\right)^k} - \cancel{\left(\tfrac{1}{3}\right)^1} - \cancel{\left(\tfrac{1}{3}\right)^2} - \cancel{\left(\tfrac{1}{3}\right)^3} - \ldots - \cancel{\left(\tfrac{1}{3}\right)^k} - \left(\tfrac{1}{3}\right)^{k+1}\right]}{1-\frac{1}{3}}$$

$$= \lim_{k\to\infty} \frac{2\left[1 - \cancel{\left(\tfrac{1}{3}\right)^{k+1}}\right]}{1-\frac{1}{3}} \quad \to 0$$

$$\sum_{n=0}^{\infty} 2\left(\tfrac{1}{3}\right)^n = \frac{2(1)}{1-\frac{1}{3}}$$

$$= 2\div\tfrac{2}{3} = 2\cdot\tfrac{3}{2} = 3$$

Reflecting on the previous examples, the following pattern emerges for $\sum_{n=0}^{\infty} a x^n$.

$$\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}$$

← only valid for small x's

The expression $\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}$ is a **power series** (and specifically a **geometric series**). It equals a number when $-1<x<1$. It does not equal a number when $|x|\ge 1$. When a series sums to a number, we say, "The series **converges**." When a series does not converge to a number, we say, "The series **diverges**."

Definition: The Geometric Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots + x^n + \ldots$$

$$= \frac{1}{1-x}, \quad -1<x<1$$

We can modify this formula to find the power series expansion of other functions.

Example 5: Show that $\frac{1}{2-x} = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1} x^n$ for $|x|<2$.

We can modify this formula to find the power series expansion of other functions.

Example 5: Show that $\frac{1}{2-x}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}x^n$ for $|x|<2$.

$$\frac{1}{2-x}=\frac{1}{2(1-\frac{x}{2})}=\frac{1/2}{1-(\frac{x}{2})}=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)\left(\frac{x}{2}\right)^n$$

$$=\sum_{n=0}^{\infty}\frac{1}{2}\cdot\left(\frac{1}{2}\right)^n\cdot x^n$$

$$=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}x^n$$

when $\left|\frac{x}{2}\right|<1$ or $|x|<2$.

recall $\frac{a}{1-x}=\sum_{n=0}^{\infty}ax^n$

$-1<x<1$

$\Leftrightarrow |x|<1$

Example 6: Find a power series expansion for $f(x)=\frac{1}{1+x^2}$. When does this series converge (equal a number for a given value of x).

$f(x)=\frac{1}{1+x^2}$

$=\frac{1}{1-(-x^2)}$

In the form: $\sum_{n=0}^{\infty}ax^n=\frac{a}{1-x}$

$\rightarrow f(x)=\sum_{n=0}^{\infty}1(-x^2)^n$

$=\sum_{n=0}^{\infty}(-1)^n\cdot x^{2n}$

common ratio.

This converges when $|-x^2|<1\Rightarrow|x^2|<1\Rightarrow|x|<1$

Example 7: Integrate the following: $\int\frac{1}{1-8x^3}dx$

$$\int\frac{1}{1-8x^3}dx$$

$$=\int\sum_{n=0}^{\infty}(8x^3)^n\,dx$$

$$=\sum_{n=0}^{\infty}\int(8x^3)^n\,dx$$

$8^n x^{3n}$

$$=\sum_{n=0}^{\infty}8^n\int x^{3n}dx$$

$$=C+\sum_{n=0}^{\infty}\frac{8^n\cdot x^{3n+1}}{3n+1}$$

Valid when

$|8x^3|<1\Rightarrow|x^3|<\frac{1}{8}$

$\Rightarrow |x|<\sqrt[3]{\frac{1}{8}}=\frac{1}{2}$

So $-\frac{1}{2}<x<\frac{1}{2}$

interval where the power series converges.

We can begin to see the three main aspects of power series come together.

1.) Finding power series representations for functions (and using them to solve questions).
2.) Determining the x values for which the work above is valid. That is, when do the series converge/diverge?
3.) Proving that this whole process is legitimate mathematics.

This last step is (mostly) beyond the scope of Highline mathematics. While we will touch on the middle step, we will leave its finer details for another course. Most of our effort will be spent on methods for finding power series.

As we assumed in the previous example, one of the qualities of power series is that they can be manipulated through addition, subtraction, multiplication, division, differentiation, and integration to find other power series.

For example, assuming the series behave nicely (converge on some neighborhood), we have:

a.) $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n x^n\right]$ (differentiate term by term)

b.) $\int\left[\sum_{n=0}^{\infty} c_n x^n\right]dx = \sum_{n=0}^{\infty}\int c_n x^n\, dx$ (integrate term by term)

Example 8: Show that: $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$

Notice

$$1)\quad \frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$$

$$\Rightarrow \int \frac{1}{(1-x)^2}dx = \frac{1}{1-x} + C$$

observe:

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}\sum_{n=0}^{\infty} x^n$$
$$= \sum_{n=0}^{\infty}\frac{d}{dx}x^n$$
$$= \sum_{n=0}^{\infty} n x^{n-1}$$
$$= 0 + 1\cdot x^0 + 2x^1 + 3x^2 + \ldots$$
$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots \quad \text{when } |x| < 1$$

Note: Thus far, we are mostly ignoring the question of when these power series are valid.

Example 9: Find a power series representation for $\ln(1-x)$.

$$\frac{d}{dx}\ln(1-x) = \frac{1}{1-x}(-1) = -\frac{1}{1-x}$$

① or $\int \frac{-1}{1-x}\,dx = \ln(1-x) + C$

$$\Rightarrow \ln(1-x) = C + \int \frac{-1}{1-x}\,dx$$

L.H.S.

$$= 0 + \sum_{n=0}^{\infty} -x^n \quad \text{R.H.S.} \qquad \text{when } -1 < x < 1 \quad \text{or} \quad \boxed{|x| < 1}$$

$\ln(2)$ $\ln(0)$

Next: find C

Let $x = 0$: L.H.S. $\ln(1) = 0 = $ R.H.S. $C + \sum -0 \Rightarrow C = 0$

Historical note: Today we can evaluate logarithms simply by pushing a calculator button. Prior to that, mathematicians looked up the log values in books. One source of the values in the books was mathematicians evaluating power series like the one above.

Example 10: Integrate $\int \tan^{-1} x\,dx$ using power series

recall:
$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

From ex. 6.
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\Rightarrow \tan^{-1}x = \int \frac{1}{1+x^2}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,dx$$

R.H.S. $= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx$

$= C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ R.H.S.

Find C
set $x = 0$
$\tan^{-1}(0) = 0 = C + \sum 0$
$C = 0.$

$$\int \tan^{-1}x\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}\,dx$$

$$= \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{2n+1}}{2n+1}\,dx$$

$$= C_1 + \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+2}}{(2n+1)(2n+2)}$$

converges when $|x| < 1$

Historical note: Earlier in calculus, we learned to integrate questions like this using integration by parts. Power series provides an alternative approach that requires no calculus skills beyond integrating/differentiating polynomial terms.

Example 11: Solve $f(x) = f'(x)$ using power series.

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$\Rightarrow f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$$

$c_0 = c_1$

$c_1 = 2c_2$

$c_2 = 3c_3$

$c_3 = 4c_4$

$\vdots$

$c_n = (n+1) c_{n+1}$

c_0

$c_1 = c_0$

$c_2 = \frac{1}{2} c_1 = \frac{1}{2} c_0$

$c_3 = \frac{1}{3} c_2 = \frac{1}{3} \cdot \frac{1}{2} c_0$

$c_4 = \frac{1}{4} c_3 = \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2} c_0$

$\vdots$

$c_n = \frac{1}{n(n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1} c_0 = \frac{1}{n!} c_0$

Thus: $f(x) = c_0 \left(1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \dots \right)$

$$= c_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$f(x) = c_0 e^x$$

Note: In this solution we took a term-by-term approach to finding the coefficients $c_0, c_1, c_2, c_3, \ldots, c_n, \ldots$

This has the same end result as finding a formula $\sum_{n=0}^{\infty} c_n x^n$, it just isn't as concise.

> **Historical note**: You may recognize the question above as a differential equation. Power series provides an extremely powerful technique for solving differential equations that can work on many many questions. It frequently isn't the fastest or cleanest approach … but it works. This is why power series have historically been the "Swiss Army Knife" of functions … they work in a wide variety of situations.