II.8 and II.9: Power Series Math 163: Calculus III (Fall 2022)

Power Series, part 1

❖ Power Series by the pictures

Intuitively, a **power series** is like an infinitely long polynomial (except that polynomials are defined so as to have finite length). Examples include:

a.) $f(x) = \frac{1}{1} + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 + \frac{1}{6}x^5 + \dots$ b.) $g(x) = \frac{1}{3}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 + \frac{1}{6}x^5 + \dots$ c.) $h(x) = \frac{1}{1} + \frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{4}x^3 + \frac{1}{5}x^4 + \frac{1}{6}x^5 + \dots$

Key idea: Working with power series $c_0 + c_1x + c_2x^2 + c_3x^3 + ... + c_nx^n + ...$ is first and foremost about finding the coefficients $c_0, c_1, c_2, c_3, ..., c_n$... Often (at least in class) we find a nice formula for the

coefficients and simply write $\sum_{i=1}^{\infty} c_i c_i$ But, either way the task at hand begins with finding coefficients.

About now, you might be wondering why anyone would care about power series. The next example

provides a graphical connection between power series and more familiar topics.

Example 1: Use a graph to explore $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ $\sum_{i=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = \frac{1}{1} + \frac{(-1)^n x^{2n}}{2}$ $\sum_{i=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{1}{24} + \frac{x^4}{24}$ $\sum_{i=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{320}$

Page 1 of 8

11.8 and 11.9: Power Series Math 163: Calculus III (Fall 2022)

Polynomial functions can be evaluated using basic operations (addition, subtraction, multiplication and division) and they can be differentiated / integrated pretty easily. But this is not the same for many other functions such as trig, exponential or logarithmic functions! It is beneficial to rewrite a function as a polynomial. This strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions values. Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.

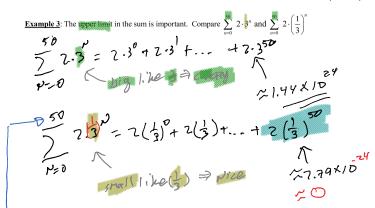
There are infinitely many power series, but some are famous enough to merit a name. The first of these is named the **geometric series**. The next few examples help us understand this very important (but basic)

We will begin with examples without x and then work our way toward actual **power series**

cample 2: Evaluate the following:
a.)
$$\sum_{n=0}^{\infty} 2 \cdot 3^n = 2 \cdot 3^n + 2 \cdot$$

b)
$$\sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{3}\right)^n = 2\left(\frac{1}{3}\right)^n + 2\left(\frac{1}{3}\right)^1 + 2\left(\frac{1}{3}\right)^2 + \dots + 2\left(\frac{1}{3}\right)^n = \frac{726}{243} \times 2.999$$

Page 2 of 8



And most interesting is when we allow the upper limit to be infinite in which case we are left with what is called an **infinite series**.

Example 4: Evaluate the infinite series
$$\sum_{n=0}^{\infty} \frac{1}{2^n} \left(\frac{1}{3} \right)^n + 2 \left($$

II.8 and II.9: Power Series Math 163: Calculus III (Fall 2022)

Reflecting on the previous examples, the following pattern emerges for $\sum_{n=0}^{\infty} a^{n}$.

The expression $\sum_{n=0}^{\infty} a x^n = \frac{a}{1-x}$ is a **power series** (and specifically a **geometric series**). It equals a number when $|x| \ge 1$. When a series sums to a number, we say, "The series **converges**." When a series does not converge to a number, we say, "The series **diverges**."

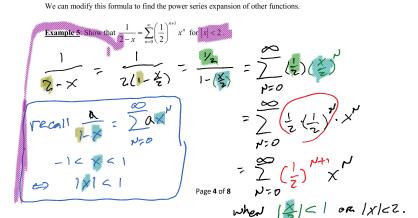
<u>Definition</u>: The Geometric Series

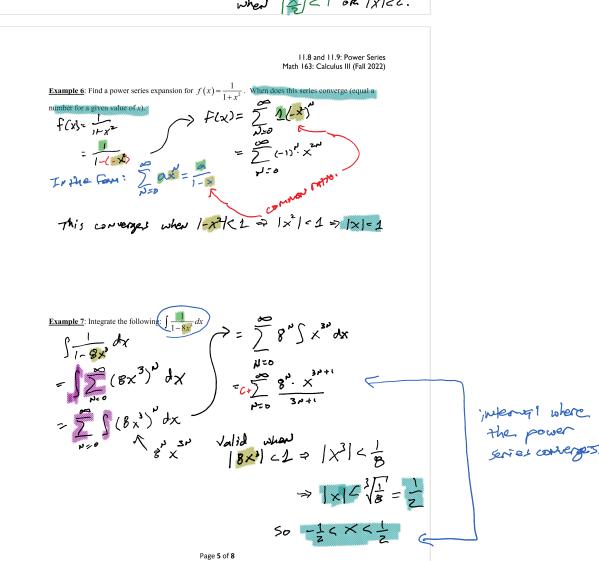
$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots + x^n + \dots$$

$$= \frac{1}{1-x}, \quad -1 < x < 1$$

We can modify this formula to find the power series expansion of other functions.

Example 5: Show that
$$\frac{1}{2-x} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} x^n$$
 for $|x| < 2$.





II.8 and II.9: Power Series Math 163: Calculus III (Fall 2022)

We can begin to see the three main aspects of power series come together.

inding power series representations for functions (and using them to solve questions).

(2.) Determining the x values for which the work above is valid. That is, when do the series converge/diverge?

3.) Proving that this whole process is legitimate mathematics.

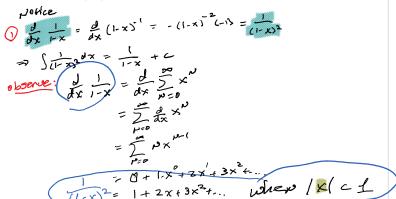
This last step is (mostly) beyond the scope of Highline mathematics. While we will touch on the middle step, we will leave its finer details for another course. Most of our effort will be spent on methods for finding power series.

As we assumed in the previous example, one of the qualities of power series is that they can be manipulated through addition, subtraction, multiplication, division, differentiation, and integration to find other power series.

For example, assuming the series behave nicely (converge on some neighborhood), we have:

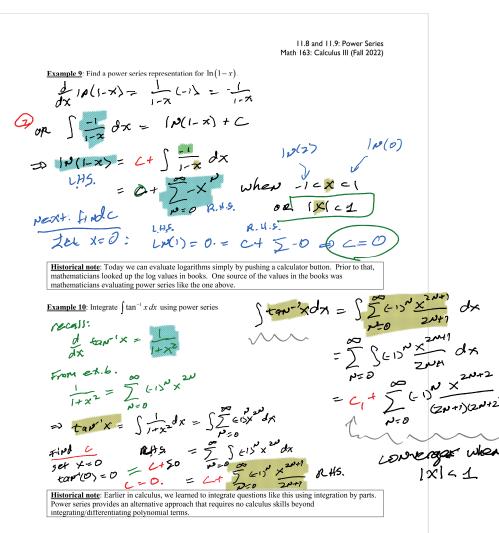
a.)
$$\frac{d}{dx} \left[\sum_{n=0}^{\infty} c_n x^n \right] = \sum_{n=0}^{\infty} \frac{d}{dx} \left[c_n x^n \right] \text{ (differentiate term by term)}$$
b.)
$$\left[\sum_{n=0}^{\infty} c_n x^n \right] dx = \sum_{n=0}^{\infty} c_n x^n dx \text{ (integrate term by term)}$$

Example 8: Show that: $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + ...$



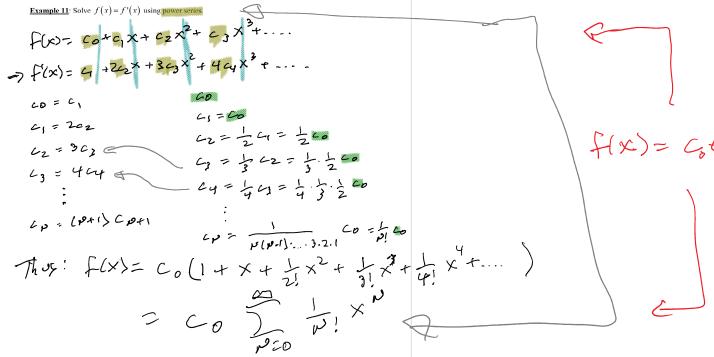
Note: Thus far, we are mostly ignoring the question of when these power series are valid

Page 6 of 8



Page **7** of **8**

II.8 and II.9: Power Series Math I63: Calculus III (Fall 2022)



Note: In this solution we took a term-by-term approach to finding the coefficients $c_a, c_1, c_2, c_3, \dots, c_n$

This has the same end result as finding a formula $\sum c_n x^n$, it just isn't as concise.

<u>Historical note</u>: You may recognize the question above as a differential equation. Power series provides an extremely powerful technique for solving differential equations that can work on many many questions. If frequently isn't the fastest or cleanest approach ... but it works. This is why power series have historically been the "Swiss Army Knife" of functions ... they work in a wide variety of situations.

Page **8** of **8**