15.1: Intro to Double Integrals

Thursday, December 1, 2022 8:47 AM

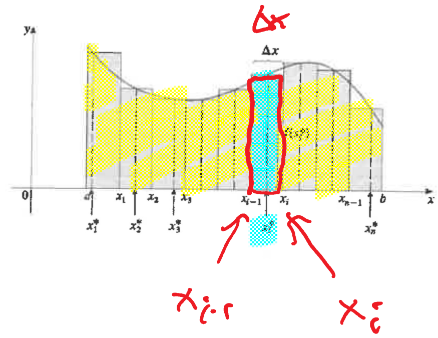
Double Integrals over Rectangles

Review of Definition of Integrals

Definition: The area A of the region S that lies under the graph of the non-negative continuous function f, between a and b, is the limit of the sum of the areas of approximating rectangles:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

where n is the number of subintervals $[x_{i-1}, x_i]$ of equal width Δx and x_i^* is called the sample point, which is any number in the subinterval $[x_{i-1}, x_i]$.

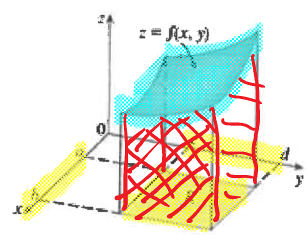


❖ **Definition of Double Integrals**

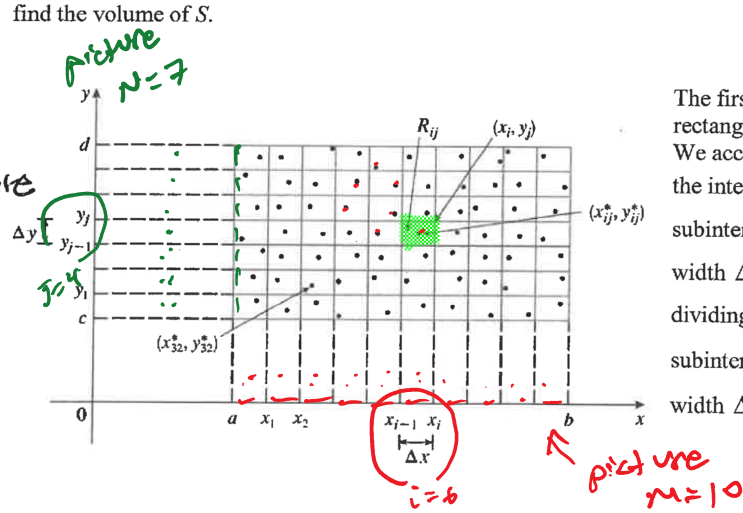
In a similar manner we consider a function f of two variables defined on a closed rectangle

$$R = [a,b]\times[\cancel{C,D}]^{c,d} = \{(x,y)\in\mathbb{R}^2 \mid a\le x\le b, c\le y\le d\}$$

And we first suppose that $f(x,y)\ge 0$. The graph of f is a surface with equation $z = f(x,y)$. Let S be the solid that lies above R and under the graph of f, that is
$S = \{(x,y,z)\in\mathbb{R}^3 \mid 0\le z\le f(x,y), (x,y)\in R\}$. Our goal is to find the volume of S.



Manipulate 16.2

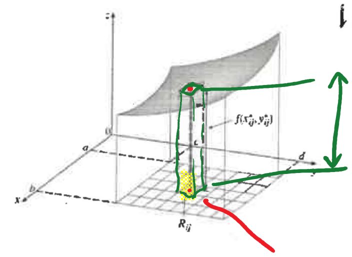


The first step is to divide the rectangle R into subrectangles. We accomplish this by dividing the interval $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b-a)/m$ and dividing the interval $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d-c)/n$.

By drawing lines parrallel to the coordinate axes through the endpoins of these ~~subitervals~~ intervals, we form the subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ each with area $\Delta A = \Delta x \Delta y$.

Manipulate 16.3

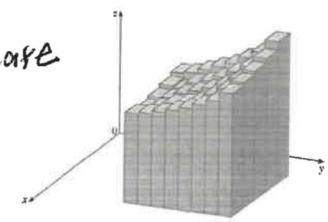
If we choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}, then we can approximate the part of S that lies above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. The volume of this box is the height of the box times the area of the base rectangle: $f(x_{ij}^*, y_{ij}^*)\Delta A$.



$f(x_{ij}^*, y_{ij}^*)$

Area $= \Delta x \Delta y$
$= \Delta A$

Manipulate 16.4



If we follow this procedure for all the rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volumne of S. $V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$. This double sum means that for each subrectangle, we evalute f at the chosen point and multiply by the area of the subrectangle, and then we add the results.

The expression $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$ is called a **double Riemann sum** and is used as an approximation to the value of the double integral. Notice how similar it is to the Riemann sum for a function of one variable.

Our intuition tells us that the approximation given becomes better as m and n become larger so as we would expect $V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$

We use this expression to define the volume of the solid S that lies under the graph of the continuous, nonnegative function f and above the rectangle R. In the case where f has both positive and negative values on R, it represents a difference of two volumes: the volume of the solid that is above R, but below f minus the volume of the solid that is below R but above f. We call this difference the **net signed volume** between R and f. Thus a positive value means that there is more volume above R than below, a negative value means that there is more volume below than above and a zero means that the two volumes are the same.

Limits of this type occur frequently, not just in finding volumes but in a variety of other situations as well – even when f is not a positive function. So we make the following definition.

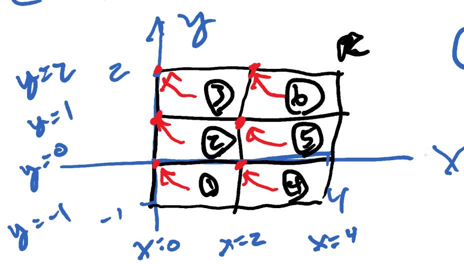
Definition: The **double integral** of f over the rectangle R is $\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^{*}, y_{ij}^{*}\right)\Delta A$ if this limit exists.

A function f is called **integrable** if the limit above exists. It is shown in courses on advanced calculus that all continuous functions are integrable.

Example 1: If $R=[0,4]\times[-1,2]$, $m=2$ and $n=3$, estimate the volume of the solid that lies between $z=1-xy^2$ and R. $f(x,y)$

a) Take the sample points to be the upper left corner of the rectangles.

① Draw R



② Area of rectangles $= 2(1) = 2$ ΔA.

③ write out the sum.

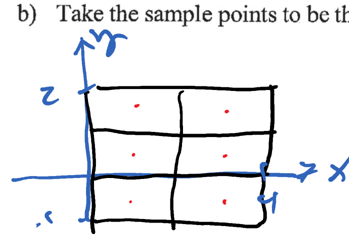
$f(0,0)\Delta A + f(0,1)\Delta A + f(0,2)\Delta A + f(2,0)\Delta A + f(2,1)\Delta A + f(2,2)\Delta A$

$= 2\left(1 + 1 + 1 + 1 + (-1) + (-7)\right)$

$= -8$ ← Approx "volume" of the solid.

recall: $z = 1 - xy^2$.

b) Take the sample points to be the midpoint of the rectangles. ΔA $z = 1 - xy^2$



volume $\approx 2\left(f(1,-\tfrac{1}{2}) + f(1,\tfrac{1}{2}) + f(1,\tfrac{3}{2}) + f(3,-\tfrac{1}{2}) + f(3,\tfrac{1}{2}) + f(3,\tfrac{3}{2})\right)$

$= 2\left(\tfrac{3}{4} + \tfrac{3}{4} + \left(-\tfrac{5}{4}\right) + \tfrac{1}{4} + \tfrac{1}{4} + \left(-\tfrac{23}{4}\right)\right) = -10$

Approx "volume" under the surface.

Key idea: A double integral represents a quantity (say volume) but does not contain any inherent mechanism for calculating that quantity. For that we will need an iterated integral.

❖ Iterated Integrals

Except in the simplest cases, it is impractical to obtain the value of a double integral from its limit definition. We now see how to evaluate double integrals by calculating two successive single integrals.

The partial derivatives of a function $f(x,y)$ are calculated by holding one of the variables fixed and differentiating with respect to the other variable. Let's consider the reverse of this process, **partial integration**. The symbols $\int_a^b f(x,y)\,dx$ and $\int_c^d f(x,y)\,dy$ denote **partial definite integrals**. The first integral, called the **partial definite integral with respect to** x, is evaluated by holding y fixed and integrate with respect to x. Similarly, the second integral, called the **partial definite integral with respect to** y, is evaluated by holding x fixed and integrate with respect to y.

Example 2: Let $f(x,y) = xy^2$ find $\int_0^1 f(x,y)\,dx$ and $\int_2^3 f(x,y)\,dy$.

$$\int_0^1 x y^2\,dx = y^2 \int_0^1 x\,dx = y^2\left[\frac{x^2}{2}\right]_0^1 = \frac{y^2}{2}$$

no x's

$$\int_2^3 x y^2\,dy = x\int_2^3 y^2\,dy = x\left[\frac{y^3}{3}\right]_2^3 = \frac{19}{3}x$$

no y's.

A partial definite integral with respect to x is a function of y and hence can be integrated with respect to y. Similarly, a partial definite integral with respect to y is a function of x which can be integrated with respect to x. This two-stage integration process is called **iterated** (or **repeated**) **integration**.

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

Example 3: Evaluate the following

a) $\int_{-1}^{1}\int_{0}^{2}\left(4+9x^2y^2\right)dydx$

$$= \int_{-1}^{1}\left[\int_0^2 (4+9x^2y^2)\,dy\right]dx$$

$$= \int_{-1}^{1}\left[4y+3x^2y^3\right]_{y=0}^{y=2} dx$$

$$= \int_{-1}^{1}(8+24x^2)\,dx$$

$$= \left[8x+8x^3\right]_{-1}^{1}$$

$$= (8+8)-(-8-8)$$

$$= 32$$ ← exact area under the surface.

b) $\int_{0}^{2}\int_{-1}^{1}\left(4+9x^2y^2\right)dxdy$

$$= \int_0^2\left[4x+3x^3y^2\right]_{x=-1}^{x=1} dy$$ manipulate 16.7

$$= \int_0^2\left[(4+3y^2)-(-4-3y^2)\right]dy$$

$$= \int_0^2 (8+6y^2)\,dy$$

$$= \left[8y+2y^3\right]_0^2$$

$$= 16+16$$

$$= 32$$

It is no accident that the two iterated integrals have the same value. Sometimes finding one is easier than finding the other.

Example 4: Evaluate $\iint_R x\cos(xy)\,dA$ where $R = [0, 1] \times \left[0, \frac{\pi}{2}\right]$ in both orders.

$$\iint_R x\cos xy\,dA = \int_0^1 \int_0^{\pi/2} x\cos(xy)\,dy\,dx$$

$$= \int_0^1 \left[\sin(xy)\right]_{y=0}^{y=\pi/2} dx$$

$$= \int_0^1 \sin\left(\frac{\pi}{2}x\right) dx$$

$$= \left[-\cos\left(\frac{\pi}{2}x\right)\cdot\frac{2}{\pi}\right]_0^1$$

$$= -\frac{2}{\pi}\Big(\underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{0} - \underbrace{\cos(0)}_{1}\Big)$$

easy

$$= \frac{2}{\pi}$$

$$\iint_R x\cos(xy)\,dA = \int_0^{\pi/2}\int_0^1 x\cos xy\,dx\,dy$$

Integration by parts $= \int_0^{\pi/2} \left[\frac{x}{y}\sin xy - \int \frac{\sin xy}{y}\,dx\right]_0^1 dy$

$u = x \qquad dv = \cos xy\,dx$

$du = dx \qquad v = \dfrac{\sin xy}{y}$

keep going?

stop.

hard.

Iterated integrals have an inbuilt order of calculation from inner integral to outer integral. Fubini's Theorem provides the basis for changing this order.

Fubini's Theorem: If f is continuous on the rectangle $R=[a,b]\times[c,d]$, then

double integral

$$\iint_R f(x,y)\,dA=\int_c^d\int_a^b f(x,y)\,dx\,dy=\int_a^b\int_c^d f(x,y)\,dy\,dx$$

iterated integrals.

More generally, this is true if we assume that f is bounded on R, f is discontnous only on a finite number of smooth curves, and the iterated integrals exist.

Example 5: Use a double integral to find the volume of the solid that is bounded above by the plane $z=4-x-y$ and below by the rectangle $R=\{(x,y)|0\le x\le 1, 0\le y\le 2\}$.

$$\iint_R (4-x-y)\,dA=\int_0^2\int_0^1(4-x-y)\,dx\,dy$$

$$=\int_0^2\left[4x-\frac{x^2}{2}-yx\right]_{x=0}^{x=1}dy$$

$$=\int_0^2\left(4-\frac{1}{2}-y\right)dy$$

$$=\left[\frac{7}{2}y-\frac{y^2}{2}\right]_0^2 \quad 5$$

exact volume.

Handy Trick: If $f(x,y)=f(x)g(y)$ and integrating over a rectangular region R then $\iint_R f(x)g(y)\,dA=\int_a^b f(x)\,dx\int_c^d g(y)\,dy$ where $R=[a,b]\times[c,d]$.

Example 6: Evaluate $\iint_R 5y^3\cos(x)\,dA$ over the rectangle $R=\left\{(x,y)\middle|0\le x\le\frac{\pi}{2}, 0\le y\le 1\right\}$.

$$=\int_0^{\pi/2}\cos x\,dx\int_0^1 5y^3\,dy$$

$$=\left[\sin x\right]_0^{\pi/2}\left[\frac{5}{4}y^4\right]_0^1$$

$$=1\cdot\frac{5}{4}$$

$$=\frac{5}{4}$$

Historical Note: Fubini's Theorem is named after Guido Fubini. Fubini was born in 1879 and lived most of his life in Italy. He worked in a wide variety of fields related to mathematical analysis (calculus and topics based upon calculus). During World War I, he shifted his work towards more applied topics, studying the accuracy of artillery fire. After the war he studied electrical circuits and acoustics. Then, in 1938 as he was nearing retirement, the Fascists adopted the anti-Jewish policies of the Nazis. As a Jew, Fubini feared for the safety of his family, and so accepted an invitation by Princeton University to teach there. He died in America four years later.

❖ **Average Value**

Previously in calculus, we saw that the average value of a function of one variable on an interval $[a,b]$ is

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Here we can define the average value of a function of two variables on a rectangle R similarly:

$$f_{ave} = \frac{1}{\text{area of } R}\iint_R f(x,y)\,dA$$

If $f(x,y) \geq 0$, we can write

$$(\text{area of } R)\times f_{ave} = \iint_R f(x,y)\,dA$$

which says that the box with base R and height f_{ave} has the same volume as the solid that lies under the graph of f.

Example 7: Find the average value of $f(x,y) = \dfrac{\tan x}{\sqrt{1-y^2}}$ over the rectangle $\left[0, \tfrac{\pi}{3}\right]\times\left[0, \tfrac{1}{2}\right]$

$$f_{ave} = \frac{1}{\text{Area of } R}\iint_R \frac{\tan x}{\sqrt{1-y^2}}\,dA$$

Area of R

$$\frac{\pi}{3}\cdot\frac{1}{2} = \frac{\pi}{6}$$

$$= \frac{6}{\pi}\int_0^{\pi/3}\tan x\,dx \int_0^{1/2}\frac{1}{\sqrt{1-y^2}}\,dy$$

$$= \frac{6}{\pi}\int_0^{\pi/3}\frac{\sin x}{\cos x}\,dx\,\left[\sin^{-1}(y)\right]_0^{1/2}$$

$$= \frac{6}{\pi}\left[-\ln(\cos x)\right]_0^{\pi/3}\left(\frac{\pi}{6} - 0\right)$$

$$= 1\left(-\ln\left(\tfrac{1}{2}\right) + \underbrace{\ln(1)}_{0}\right)$$

$$= \ln 2$$

Hence f_{ave} is $\ln 2$.