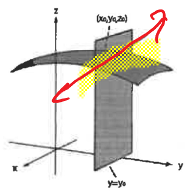
14.6: Directional Derivatives

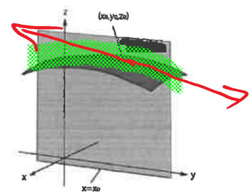
Monday, November 28, 2022 2:36 PM

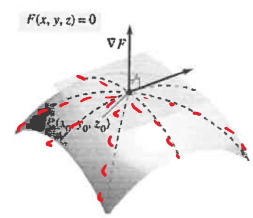
Directional Derivatives and the Gradient Vector

❖ **Directional Derivatives**

In a previous lesson we talked about the tangent plane. To come up with the equation of tangent plane we considered two curves on the surface, C_1 : the intersection of S and the plane $y = y_0$ and C_2 : the intersection of S and the plane $x = x_0$. Then we found the tangent line to these curves. But why did we pick those?

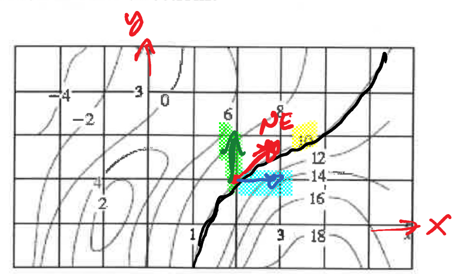






I told you we pick them because they are easier. Well the real reason is that we pick them because we only knew how to take the partial derivative of a function with respect to our main axis so with respect to x and y. In this section we will define partial derivatives of a function in other directions.

Recall the first homework problem you had in 14.3. The question asked you to find $f_x(2,1)$ and $f_y(2,1)$. But what is the meaning of these notations? Let's give the function a story. Let's say $z = f(x, y)$ is representing the surface on a 3D mountain. So z is the height. $f_x(2,1)$ gives you the rate of change of height in direction of x that means how fast or slow your height (slope) changes as you walk in the direction of positive x-axis, passing through the point $(2,1,10)$.



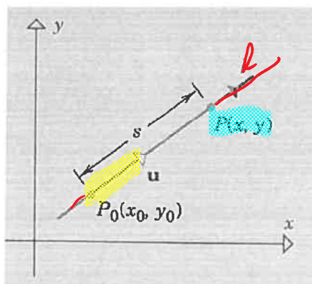
With our current knowledge we can only find the change in height, walking in the direction of positive x-axis (East) and walking in the direction of positive y-axis (North). But there are so many other directions! What if I want to walk in the direction NE? Directional derivative will help us with that. We generally use a unit vector to show the direction because we don't want the length of our vector influence our perspective of rate of change. To understand that think about running verses walking in direction of NE. That may influence your perspective of how fast the height of the mountain changes (speed of one unit of distance per unit of time)!

Let $z = f(x, y)$. We need to find the partial derivative of this function in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ that lives on the xy-plane at the point (x_0, y_0, z_0). Let l be the line in the xy-plane that is parallel to $\vec{u}$ and passes through $P_0(x_0, y_0)$. This line can be represented by parametric equations:

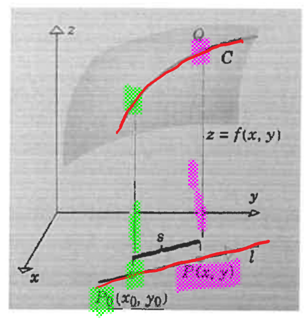
$$x = x_0 + ta \qquad y = y_0 + tb$$

where t is an arc-length parameter with its reference point at $P_0(x_0, y_0)$, and the positive direction is in the direction of $\vec{u}$. As t increases, the point $P(x, y)$ moves in the direction of $\vec{u}$ along l, and a companion point Q with z-coordinate $z = f(x, y) = f(x_0 + ta, y_0 + tb)$ moves directly above (or below) along the surface, tracing out a curve C.

2D



3D



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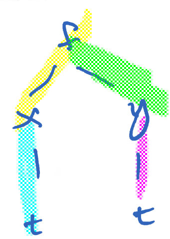
The rate of change of z with respect to t can be calculated using the chain rule:

$$\frac{dz}{dt} = f_x(x, y)\frac{dx}{dt} + f_y(x, y)\frac{dy}{dt}$$



Note that since $x = x_0 + ta$ $y = y_0 + tb$ then $\frac{dx}{dt} = a$ $\frac{dy}{dt} = b$ so:

$$\frac{dz}{dt} = f_x(x, y)a + f_y(x, y)b$$

where x and y are expressed in terms of t. But $P_0(x_0, y_0)$ is the point on l corresponding to $t = 0$:

$$\left.\frac{dz}{dt}\right|_{t=0} = f_x(x_0, y_0)a + f_y(x_0, y_0)b$$

This quantity is called the **directional derivative** of f at (x_0, y_0) in the direction of $\vec{u}$. We can use other notations such as $D_{\vec{u}} f(x_0, y_0)$.

Note that:

- If $\vec{u}$ is in the direction of the positive x-axis then $\vec{u} = \vec{i} = \langle 1,0 \rangle$ so we get $\left.\frac{dz}{dt}\right|_{t=0} = f_x(x_0, y_0)$ and
- If $\vec{u}$ is in the direction of the positive y-axis then $\vec{u} = \vec{j} = \langle 0,1 \rangle$ so we get $\left.\frac{dz}{dt}\right|_{t=0} = f_y(x_0, y_0)$

So the partial derivatives with respect to x and y are just special directional derivatives.

Theorem: If f is a differentiable function of x and y, then f has a **directional derivative** in the direction of any unit vector $\vec{u} = \langle a,b \rangle$ and $D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b$.

Example 1: Find the directional derivative of $f(x,y) = 3x^2 y$ at the point $(1,2)$ in the direction of the vector $\vec{v} = 3\vec{i} + 4\vec{j}$.

① unit vector parallel to $\vec{v}$.

$\vec{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

$\uparrow a \quad \uparrow b$

② $f_x(1,2) = 6xy|_{(1,2)}$ 12

③ $f_y(1,2) = 3x^2|_{(1,2)}$ 3

$D_{\vec{u}} f(1,2) = 12\left(\frac{3}{5}\right) + 3\left(\frac{4}{5}\right)$

$= \frac{48}{5}$

Manipulate
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It is worth noting that reversing the direction of $\vec{u}$ reverses the sign of the directional derivative:

$$\begin{aligned} D_{-\vec{u}} f(x_0, y_0) &= f_x(x_0, y_0)(-a) + f_y(x_0, y_0)(-b) \\ &= -\left[f_x(x_0, y_0)a + f_y(x_0, y_0)b \right] \\ &= -D_{\vec{u}} f(x_0, y_0) \end{aligned}$$

❖ *The Gradient Vector*

The directional derivative formula can be expressed in the form of a dot product:

Δ delta

∇ Nabla

$$
\begin{aligned}
D_{\vec{u}} f(x,y) &= f_x(x,y)a + f_y(x,y)b \\
&= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a,b \rangle \\
&= \langle f_x(x,y), f_y(x,y) \rangle \cdot \vec{u}
\end{aligned}
$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well. So we give it a special name "the gradient of f" and a special notation: **grad** f or ∇f, which is read "del f."

Definition: If f is a function of two variable x and y, then the gradient of f is the vector function ∇f defined by $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \dfrac{\partial f}{\partial x}\vec{i} + \dfrac{\partial f}{\partial y}\vec{j}$.

manipulate
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With the gradient notation, directional derivative can be written as: $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$ where $\vec{u}$ is a unit vector.

Example 2: Find the gradient of $f(x,y) = 3x^2 y$ at the point $(1,2)$ and use it to calculate the directional derivative of f at $(1,2)$ in the direction of the vector $\vec{v} = 3\vec{i} + 4\vec{j}$.

① find $\hat{u} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$

② $\nabla f(x,y) = \langle 6xy, 3x^2 \rangle \Big|_{(1,2)}$ $\langle 12, 3 \rangle$

Thus $D_{\vec{u}} f(1,2) = \langle 12, 3 \rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle = \frac{48}{5}$

❖ *Functions of Three Variables*

For functions of three variables we can define directional derivatives in a similar manner. Again $D_{\vec{u}} f(x,y,z)$ can be interpreted as the rate of change of the function in the direction of a unit vector $\vec{u}$ and $D_{\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot \vec{u}$

Example 3: Find the gradient of $f(x,y,z)=x^2y-yz^3+z$ at the point $(1,-2,0)$ and use it to calculate the directional derivative of f at $(1,-2,0)$ in the direction of the vector $\vec{v}=2\vec{i}+\vec{j}-2\vec{k}$.

① find $\vec{u}=\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \rangle$

② find $\nabla f(x,y,z)=\langle 2xy, x^2-z^3, -3yz^2+1 \rangle \Big|_{(1,-2,0)} \langle -4,1,1 \rangle$

Thus $D_{\vec{u}} f(1,-2,0)=\langle -4,1,1 \rangle \cdot \langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \rangle = 3$

Thus the slope f in the direction of $\vec{v}$ @ point $(1,-2,0)$ is 3.

❖ ***Significance of the Gradient Vector***

The gradient is not just a notational device to simplify the formula for the directional derivative; the length and direction of the gradient ∇f provides important information about the function. For example, suppose $z=f(x,y)$ is representing the surface of a mountain, that is z is the height at (x,y). If we want to know in which direction f is steepest and what that rate of change is, we need look no farther than the gradient vector. Let's see how!

- Write the dot product for directional derivative of f in the direction of $\vec{u}$: $D_{\vec{u}} f = \nabla f \cdot \vec{u}$
- Rewrite this dot product using θ, the angle between ∇f and $\vec{u}$: $\Rightarrow$ $= |\nabla f||\vec{u}|\cos\theta$
- What is magnitude of $\vec{u}$? $\underbrace{|\vec{u}|}_{1}$
- Rewrite it: $\Rightarrow$ $D_{\vec{u}} f = |\nabla f|\cos\theta$
- What is maximum value of $\cos\theta$? max cosine is 1
- When does it happen? when $\theta=0$
- If $\theta=0$, what is the relationship between ∇f and $\vec{u}$? they are in the same direction.
- What can we conclude? The gradient is in the direction of max directional derivative.

Theorem: Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\vec{u}} f$ is $|\nabla f|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector ∇f.

Example 4: Go back to example 3 and find the maximum rate of increase of f at P.

recall: $f(x,y,z) = x^2y - yz^3 + z$ and $P(1,-2,0)$.

$\nabla f(x,y,z) = \langle 2xy, x^2 - z^3, -3yz^2 + 1\rangle \Big|_{(1,-2,0)} \langle -4, 1, 1\rangle$

$\Rightarrow |\nabla f| = |\langle -4, 1, 1\rangle| = \sqrt{18}$ ← max r.o.c.

← less than $|\nabla f|$ in another direction r.o.c. = 3

Example 5: For the function $f(x,y) = x^2e^y$, find the maximum value of the directional derivative at $(-2,0)$, and give a unit vector in the direction in which the maximum value occurs.

$\nabla f = \langle 2xe^y, x^2e^y\rangle \Big|_{(-2,0)}$ $\langle -4, 4\rangle$ $\vec{v}$.

max r.o.c. $|\nabla f| = \sqrt{16+16} = \sqrt{32}$

$\langle -\frac{4}{\sqrt{32}}, \frac{4}{\sqrt{32}}\rangle = \vec{u}$

direction of max increase: $\Rightarrow \vec{u} = \langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

❖ ***The Gradient Vector and Contour Plots***

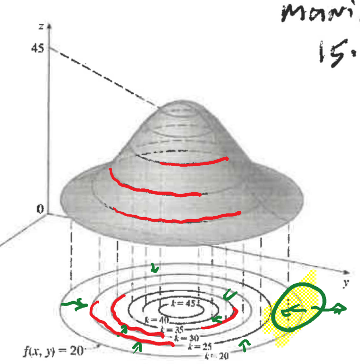
manipulate 15.51

Question: What happens to the height of a function (altitude) as you move along a level curve?

Nothing. The height is constant.

Question: Where is the gradient on a contour plot?

perpendicular to level curves and in the direction of steepest ascent (up).

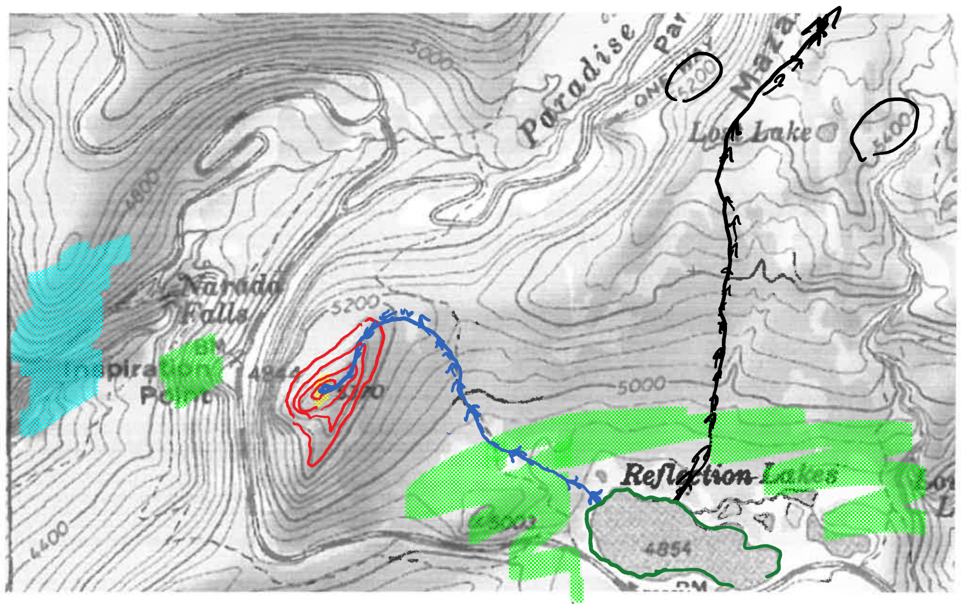


From this we can make the connection that f has maximum rate of increase in the direction of the gradient, maximum decrease in the opposite direction, and no change when orthogonal to the gradient (along the contour line or level surface).

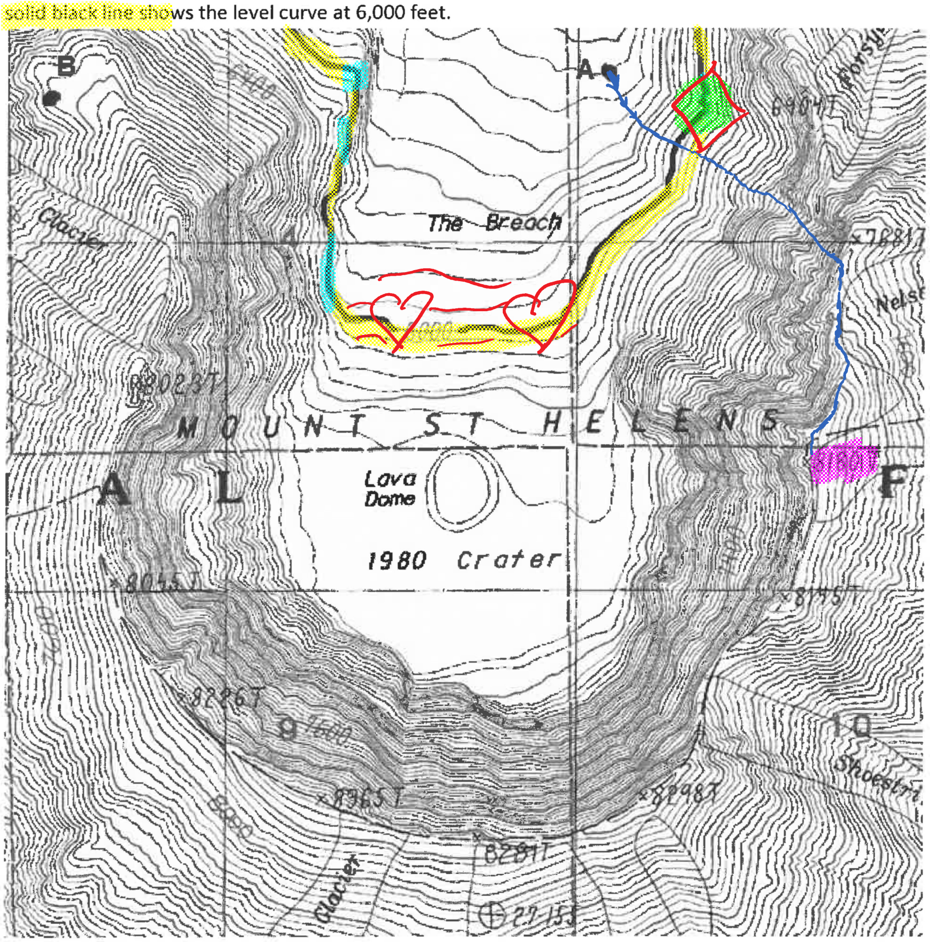
Example 6a: Consider a contour plot (topographical map) of Des Moines near Highline College.

No contours.
In the flat valley.

Example 6b: Think about peaks, ridges, valleys, and the path of steepest ascent at Mt Rainier.



Example 6c: Consider the contour plot (topographical map) of the crater of Mt Saint Helens where $z = f(x, y)$ gives the altitude (in feet) at point (x, y) where *x* and *y* have the traditional orientation. The solid black line shows the level curve at 6,000 feet.



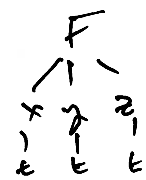
a.) On the contour plot, clearly mark with a diamond ◆ the point(s) of the level curve $f(x, y) = 6{,}000$ at which $f_x > 0$ and $f_y = 0$.

b.) On the contour plot, clearly mark with a heart ♥ the point(s) of the level curve $f(x, y) = 6{,}000$ at which the slope is shallowest ($|\nabla f|$ is small).

c.) Beginning at point ***A***, clearly sketch the path of steepest ascent.

Our last objective in this section is to establish a geometric relationship between the level surfaces and the gradient of a function of three variables.

Suppose we have a function $w = F(x,y,z)$ which lives in 4D. If we let k represent a constant then $F(x,y,z)=k$ is a level surface of your function which lives in 3D. Call this surface S. Let C be a curve laying on S. Since C is a space curve, we can define it by vector function $\vec{r}(t)=\langle x(t), y(t), z(t)\rangle$ Since C is on S we can write $F(x(t), y(t), z(t)) = k$. Assuming everything is differentiable, we can differentiate both sides of this equation:



$$\frac{\partial F}{\partial x}\frac{dx}{dt}+\frac{\partial F}{\partial y}\frac{dy}{dt}+\frac{\partial F}{\partial z}\frac{dz}{dt}=0$$

Note that $\nabla F = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right\rangle$ and $\vec{r}\,'(t)=\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle$ so this equation can be written as:

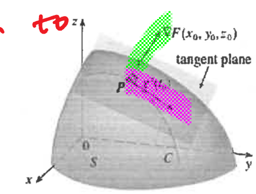
$\nabla F \cdot \vec{r}\,'(t) = 0$ which means $\nabla F \perp \vec{r}\,'(t)$

for all t

Let $P(x_0, y_0, z_0)$ be a point on C with position vector $\vec{r}(t_0)=\langle x_0, y_0, z_0\rangle$. Then:

$$\nabla F(x_0, y_0, z_0)\cdot \vec{r}\,'(t_0)=0$$

also true for t_0



This equation says that **the gradient vector at P, $\nabla f(x_0, y_0, z_0)$, is perpendicular to the tangent vector $\vec{r}\,'(t_0)$ to any curve C on S that passes through P.**

If $\nabla f(x_0, y_0, z_0)\neq 0$, we can define the **tangent plane to the level surface** $F(x,y,z)=k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla f(x_0, y_0, z_0)$. And its equation would be:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane. Hence its direction vector will be $\nabla f(x_0, y_0, z_0)$. The symmetric equation of this line is:

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)}=\frac{y-y_0}{F_y(x_0,y_0,z_0)}=\frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

Example 7: Find the equation of the plane that is tangent to the ellipsoid $\underbrace{x^2+4y^2+z^2}_{F(x,y,z)}=18$ at the point $P(1,2,-1)$. What is the equation of the normal line to this ellipsoid at P?

$\nabla F = \langle 2x, 8y, 2z \rangle \Big|_{(1,2,-1)}$ $\langle 2, 16, -2 \rangle$

plane: $2(x-1)+16(y-2)-2(z+1)=0$

normal line: $\dfrac{x-1}{2} = \dfrac{y-2}{16} = \dfrac{z+1}{-2}$