# 14.4: Tangent Planes

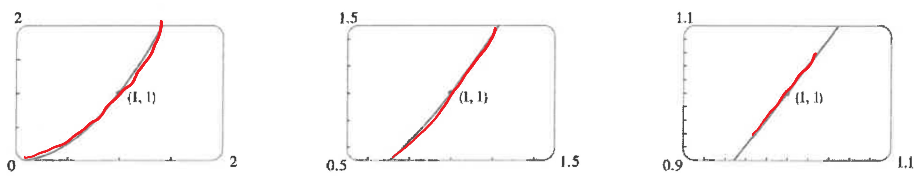
Friday, November 11, 2022 9:42 AM



***Tangent Planes and Linear Approximations***

❖ **Review**

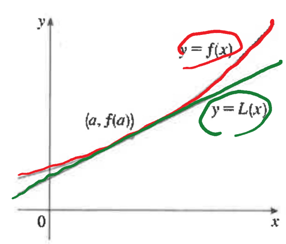
Recall from when you were first learning calculus that we sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, then curve looks almost like a straight line. For example, in the pictures below, we zoom in on the curve of $y = x^2$ near $x = 1$. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.




Later, we build upon the idea of a tangent line in order to approximate functions. The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of *f*. So we settle for the easily computed values of the linear function *L* whose graph is the tangent line of *f* at $(a, f(a))$.

In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when *x* is near *a*.

An equation of this tangent line is $y = f(a) + f'(a)(x-a)$ and the approximation $f(x) \approx f(a) + f'(a)(x-a)$ is called the **linear approximation** or **tangent line approximation** of *f* at *a*. The linear function whose graph is this tangent line, that is, $L(x) = f(a) + f'(a)(x-a)$.




To make sure we understand this, let's look at an example!

**Example 1**: Find the linearization of the function $f(x) = \sqrt{x}$ at $a = 4$ and use it to approximate $\sqrt{4.1}$.

point: $(4, \sqrt{4}) = (4, 2)$

slope: $f'(4) = \frac{1}{2\sqrt{x}}\Big|_{x=4}$ $\frac{1}{4}$

So $L(x) = 2 + \frac{1}{4}(x - 4)$

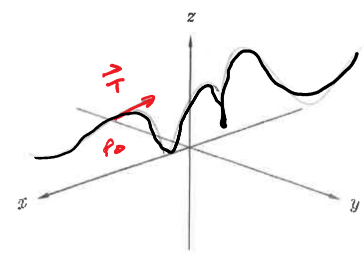


And $\sqrt{4.1} \approx L(4.1) = 2 + \frac{1}{4}(4.1 - 4) = 2.025.$

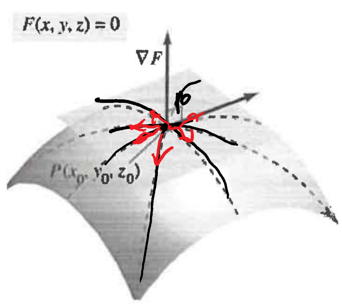
In this section we develop similar ideas in 3D. As we zoom in toward a point on a surface that is the graph of a differentiable function of two variables, the surface looks more and more like its tangent plane. We can approximate the function by a linear function of two variables.

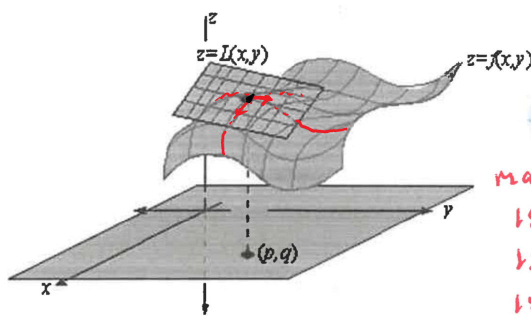
❖ **Tangent Plane**

Recall that if $C$ is a smooth parametric curve in 3D, then the tangent line to $C$ at a point $P_0$ is the line through $P_0$ along the unit tangent vector to $C$ at $P_0$.




The concept of a tangent plane builds on this definition. If $P_0$ is a point on surface $S$, and if the tangent lines at $P_0$ to all smooth curves that pass through $P_0$ and lie on the surface $S$ all lie in a common plane, then we shall regard that plane to be the **tangent plane** to the surface $S$ at $P_0$.




manipulates
15.59
15.62
15.64

Definition: We say that $z = f(x, y)$ is **differentiable** at $(x_0, y_0)$ if $f_x$ and $f_y$ exist near $(x_0, y_0)$ and continuous at $(x_0, y_0)$.

Logic questions (just for fun):

- Is it possible for a function $z = f(x, y)$ to be differentiable at $(x_0, y_0)$ even though $f_x$ and $f_y$ do not exist at $(x_0, y_0)$? No. $f_x, f_y$ are continuous @ $(x_0, y_0)$.
- Is it possible for a function $z = f(x, y)$ to be non-differentiable at $(x_0, y_0)$ even though $f_x$ and $f_y$ exist at $(x_0, y_0)$? yes. $f_x, f_y$ could exist at $(x_0, y_0)$, but not in the nearby area!!

The tangent plane to the surface $S$ of function $z = f(x,y)$ at $P_0(x_0, y_0, z_0)$ exists if and only if $z = f(x,y)$ is **differentiable** at $(x_0, y_0)$.

But how do we find the tangent plane?

Suppose we are given a surface $z = f(x,y)$ and asked to find the tangent plane at point $P_0(x_0, y_0, z_0)$. We recall that finding the equation of a plane requires that we know a point (which we already have) and a normal vector.

To find the normal vector, notice that at a given point the partial derivatives point along the plane in the $x$ and $y$ directions.

- In the $x$ direction: The slope is $f_x(x_0, y_0)$ and the vector is $\langle 1, 0, f_x(x_0, y_0)\rangle$
- In the $y$ direction: The slope is $f_y(x_0, y_0)$ and the vector is $\langle 0, 1, f_y(x_0, y_0)\rangle$
- This means the normal vector to the plane at point $P_0$ is $\langle -f_x, -f_y, 1\rangle$.

Recall: The **scalar equation of the plane** through the point $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Tangent Plane: $-f_x(x-x_0) - f_y(y-y_0) + 1(z-z_0) = 0$

which is better written as: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Thus the equation of the tangent plane is: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

**Example 2**: Find the equation of the tangent plane to the surface $f(x,y) = x^3 y$ at the point $(2,1,8)$.

$f_x(x,y) = 3x^2 y \Big|_{(2,1)}$ 12

$f_y(x,y) = x^3 \Big|_{(2,1)}$ 8

TANGENT PLANE!

And the plane is: $z - 8 = 12(x-2) + 8(y-1)$

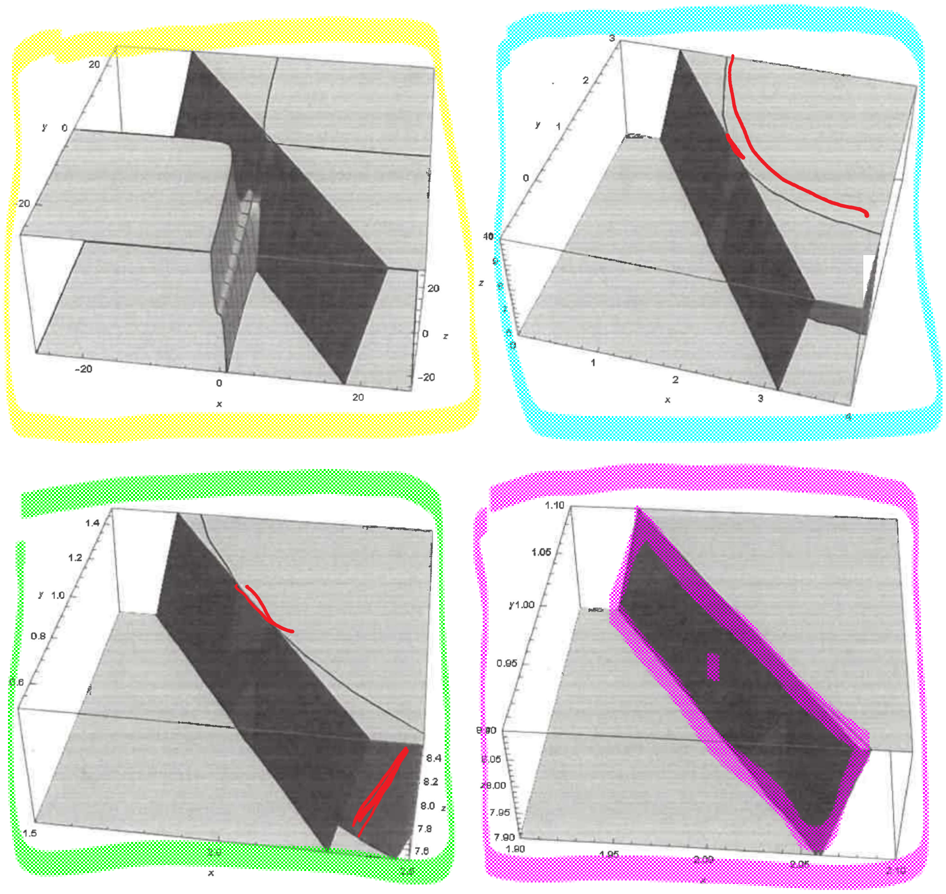
Explore: Now let's look at $f(x,y)=x^2y$ with graphing software using window $[-27,27]\times[-27,27]\times[-27,27]$...crazy shape! Hard to see where the plane and the surface really intersect. But we know the point is $(2,1,8)$. Let's get close by using these viewing windows:

$[0,4]\times[-1,3]\times[6,10]$

$[1.5,2.5]\times[.5,1.5]\times[7.5,8.5]$

$[1.9,2.1]\times[.9,1.1]\times[7.9,8.1]$

manipulate
14.4 scratch



Notice that the close we get to the point, the flatter the graph of your surface become and closer to the tangent plane! This is what we will use to do linear approximation in 3D.

❖ **Linear Approximation / Tangent Plane Approximation**

In the previous example, we found the equation of tangent plane to the function $f(x,y)=x^3y$ at the point $(2,1,8)$ to be $z=\cancel{12x+8y+8}$ $12(x-2)+8(y-1)+8$. This can be written as $L(x,y)=\cancel{12x+8y+8}$ $12(x-2)+8(y-1)+8$. This function is called the **linearization** of $f$ at $(2,1)$. The approximation $f(x,y)\approx\cancel{12x+8y+8}$ $12(x-2)+8(y-1)+8$ is called the **linear approximation** or **tangent plane approximation** of $f$ at $(2,1)$.

Approximation $\qquad$ exact.

To see how this works, find $L(1.9,.89)$ and $f(1.9,0.89)$.

$f(1.9, 0.89) = (1.9)^3(0.89) = 6.10451$ exact.

$L(1.9, 0.89) = 12(1.9-2) + 8(0.89-1)+8$

$= 12(-0.1) + 8(-0.11)+8 = 5.92$ Approx

Note that if we look at a point farther from $(2,1)$, for example $(3,2)$, we no longer get a good approximation.

Now let's find a general formula for **linearization** of $f$ at $(a,b)$:

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Since $z=f(x,y)$, $z_0=f(x_0,y_0)$. Consider the point $(a,b)$ in 3D: $(a,b,f(a,b))$

$$z-f(a,b)=f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

$$z=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

Definition: We say that the **linear approximation** or **tangent plane approximation** of $f$ at $(a,b)$:

$$f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

Note: $f$ is called differentiable at $(a,b)$ if both $f_x(a,b)$ and $f_y(a,b)$ exist and the linearization of $f$ at $(a,b)$ closely approximates $f(x,y)$ when $(x,y)$ is sufficiently close to $(a,b)$.

**Example 3**: Show that $f(x,y)=x^2e^y$ is differentiable at $(1,0)$. Find its linearization (Approx) and use it to approximate $z=f(1.05,-0.01)$.

$f(1,0) = 1\cdot 1 = 1$

$f_x(x,y) = 2xe^y \big|_{(1,0)}$ $2$

$f_y(x,y) = x^2e^y \big|_{(1,0)}$ $1$

$L(x,y) = 2(x-1) + 1(y-0) + 1$

$\to L(1.05,-0.01)$
$= 2(0.05) + 1(-0.01) + 1$
$= 1.09$ Approx

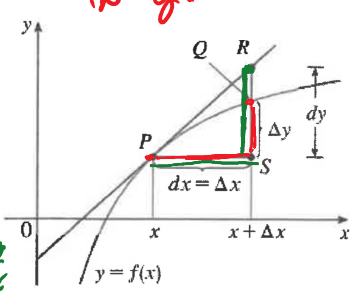
check
$f(1.05,-0.01)$
$= 1.09153$ exact

❖ ***Differentials***

Review of Differentials: Recall that if $y=f(x)$ is a differentiable function, then we defined the **differential** of $x$, defined by $dx$, is an independent variable (can be any real number). The **differential** of $y$, defined by $dy$, is then dependent on $dx$ and can be defined by equation: $dy=f'(x)\,dx$.

$\Delta y$ = exact change in $y$.

To understand the geometric meaning of this, consider the given graph:




$P$ and $Q$ are on the ~~cure~~ curve $y=f(x)$. They are $\Delta x$ away from each other. Then $\Delta y$ is the change between their $y$ values.

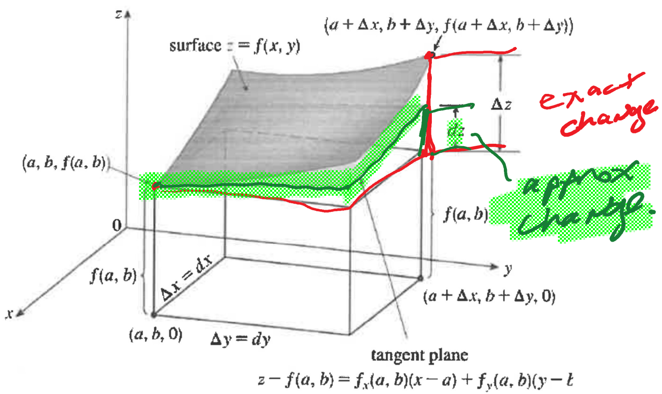
$R$ is on the tangent line, $\Delta x$ away from $P$ and the change between their $y$ values is $dy$. $dy$ = approx change in $y$.

- What is the slope of this tangent line, using rise over run? – $\frac{dy}{dx}$
- What is the slope of this tangent line, using concept of calculus? – $f'(x)$.

Therefore $dy$ represents the amount that the tangent line rises or falls (the change in linearization). In the other hand $\Delta y$ represents the amount that the curve rises or falls.

Now let's take this to 3D.

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** $dx$ and $dy$ to be independent variables (that is, they can be given any values. Then the **differential** $dz$, also called the **total differential**, is defined by:




$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$

$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$\partial$ $d$

This may look intimidating, but it is actually straight-forward. The total differential is the change in height on a tangent plane.

In words: $\begin{pmatrix}\text{approximate}\\ \text{change in } z\end{pmatrix} = \begin{pmatrix}\text{slope in}\\ x\text{ direction}\end{pmatrix}\begin{pmatrix}\text{change}\\ \text{in } x\end{pmatrix} + \begin{pmatrix}\text{slope in}\\ y\text{ direction}\end{pmatrix}\begin{pmatrix}\text{change}\\ \text{in } y\end{pmatrix}$

Remember:

$dz \leftarrow$ approximate change in height $\leftarrow$ tangent plane

$\Delta z \leftarrow$ exact change in height of the surface $\leftarrow$ fct value in 2 places.

**Example 4**: Let $f(x, y) = \sqrt{x^2 + y^2}$.

a) Find the differential $dz$.

$$dz = \frac{x}{\sqrt{x^2+y^2}}dx + \frac{y}{\sqrt{x^2+y^2}}dy$$

b) Use total differential to approximate the change in this function as $(x, y)$ varies from $(3, 4)$ to $(3.04, 3.98)$.

$dx = 0.04$
$dy = -0.02$
And $(x, y) = (3, 4)$

$$dz = \frac{3}{5}(0.04) + \frac{4}{5}(-0.02) = 0.008$$

Approx change.

c) Is the value you found in part (b.) close enough to the actual value of the fall/rise of the curve at $(3.04, 3.98)$ compare to $(3, 4)$?

find $\Delta z = f(3.04, 3.98) - f(3, 4) = 0.00819 \leftarrow$ exact change

❖ ***Functions of Three or More Variables***

Linear approximations, differentiability, and differentials can be defined in a similar manner for functions of more than two variables.

In particular, if $w = f(x, y, z)$, then the **differential** $dw$ is given by $dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$

**Example 5**: Find the differential of the function $L = xze^{-y^2-z^2}$

$$dL = ze^{-y^2-z^2}dx + xz(-2y)e^{-y^2-z^2}dy + \left(xe^{-y^2-z^2} + xz(-2z)e^{-y^2-z^2}\right)dz$$