14.1: Functions of Several Variables

Thursday, November 10, 2022 8:09 PM

Functions of Several Variables

So far we have dealt with the calculus of functions of a single variable. But, in the real world, physical quantities often depend on two or more variables. So now we turn our attention to functions of several variables.

❖ **Functions of Two or More Variables and Their Domains**

A function of two or more variables is a rule assigning a real number to every <u>point</u> in its domain. So, the domain is a subset of $\mathbb{R}^n$ and the range is a subset of $\mathbb{R}$.

There are many familiar formulas that depends on two or more variables. For example,

- Area of a triangle: $A = \frac{1}{2}bh$
- Volume of a box: $v = lwh$
- Average of a list of n values: $\bar{x} = (x_1 + x_2 + \ldots + x_n)/n$

We use the following notation:

$$y = f(x)$$
$$z = f(x, y)$$
$$w = f(x, y, z)$$
$$\ldots$$
$$u = f(x_1, x_2, \ldots, x_n)$$

<u>Definition</u>: A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is $\{f(x, y) \mid (x, y) \in D\}$.

<u>Domain</u>: If a function is given by a formula and no domain is specified, then the domain is the set of all points for which the given expression is a well-defined real number (no zero in the denominator, no negative under radical of even index, no zero or negative when taking logs,…)

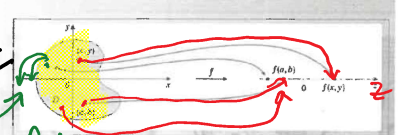
FUNCTIONS OF TWO VARIABLES

One way of visualizing such a function is by means of an arrow diagram, where the domain D is represented as a subset of the *xy*-plane.

see Manipulate 15.3

Caution

bad: not a fct



Example 1: Evaluate $f(x,y)=3x^2\sqrt{y}-1$ at the following:

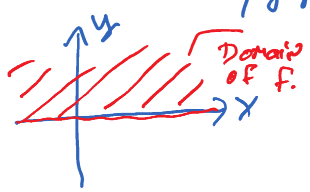
a) $(1,4)$ find $f(1,4)=3(1)^2\sqrt{4}-1=5$

point $(1,4,5)$ is on the surface/graph.

b) What is the domain of this function? Can you graph it?

$f(x,y)=3x^2\sqrt{y}-1$

Domain: $y \geq 0$.



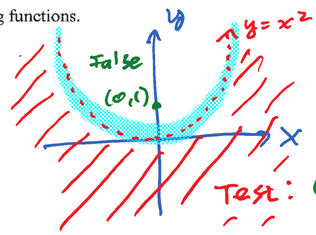
Example 2: Sketch the domain of the following functions.

a) $f(x,y)=\ln(x^2-y)$

Domain: $x^2-y>0$

$\Rightarrow x^2>y$

or $y<x^2$



Test: $(0,1)$

$1<0^2$

False

b) $f(x,y,z)=\sqrt{1-x^2-y^2-z^2}$

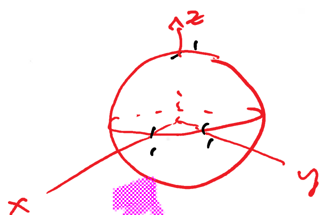
Domain: $1-x^2-y^2-z^2 \geq 0$

$\Rightarrow 1 \geq x^2+y^2+z^2$

Notice: $1=x^2+y^2+z^2$

is sphere w/ radius

1 & centered @ origin.



not to scale.

~~tennis~~ pool

solid baseball

~~soccer football~~

❖ Graph of Multivariable Functions

Recall that for a function f of one variable, the graph of $f(x)$ was defined to be the collection of all the points (x, y) where $y = f(x)$. Similarly, the graph of $f(x, y)$ is defined to be the collection of all the points (x, y, z) where $z = f(x, y)$. Previous examples we have seen of this include planes and quadric surfaces in three dimensions. (Not lines).

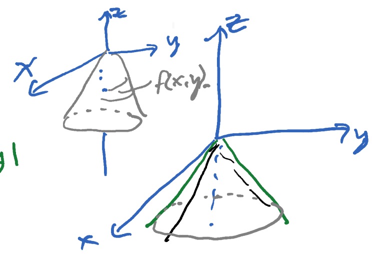
Definition: If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Example 3: Sketch $f(x, y) = -\sqrt{x^2 + y^2}$

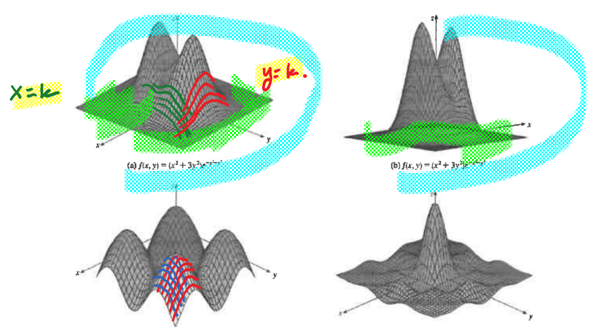
$z = f(x,y) = -\sqrt{x^2+y^2}$

If $x=0$: $z = -\sqrt{y^2} = -|y|$

If $y=0$: $z = -\sqrt{x^2} = -|x|$



The surfaces we studied previously are "easy" to graph by hand but many others are not! There are computer programs that are capable of graphing functions of two variables – surfaces in 3D. In most programs, traces in the vertical planes $x = k$ and $y = k$ are drawn for equally spaced values of k.



(a) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$ (b) $f(x, y) = (x^2 + 3y^2)e^{-x^2-y^2}$

(c) $f(x, y) = \sin x + \sin y$ (d) $f(x, y) = \frac{\sin x \sin y}{xy}$

These figures are an example of computer-generated graphs. Note that (a) and (b) are the same function from two different perspective. You can see that the graph is becoming flat as you move away from the origin. This is because $e^{-x^2-y^2}$ approaches zero ($z = 0$ aka the xy-plane) as x and/or y become larger.

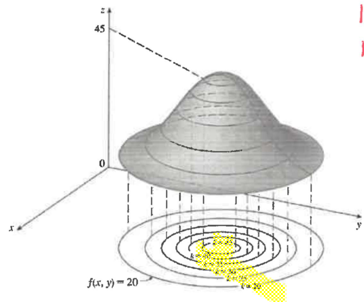
❖ Level Curves

Functions of two variables can be represented as surfaces, and can be described in two dimensions by contour maps and horizontal traces which are called **level curves**.

Definition: The **level curves** of a function *f* of two variables are the curves with equations $f(x,y)=k$, where *k* is a constant (in the range of *f*).

A level curve $f(x,y)=k$ is the set of all points in the domain of *f* at which *f* takes on a given value *k*. In other words, it shows where the graph of *f* has height *k*.

In the picture, you can see the relation between level curves and horizontal traces. The level curves $f(x,y)=k$ are just the traces of the graph of *f* in the horizontal plane $z=k$ projected down to the *xy*-plane. So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph. The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.



see manipulates
15.10
15.12
15.13
15.14b
15.16

Note: A graph made up of level curves is called a **contour plot**. Outside of math, perhaps you have seen a **topographical map**. This is (roughly) the same thing.

Important detail: When constructing a contour plot, the level curves should be equally spaced. For example $k=-4,0,4,8,12,16,\ldots$ or $k=1000,1050,1100,1150,\ldots$. In the picture above, the level curves are graphed for $k=20,25,30,35,40,45$

Example 4: Recall the function in the previous example, $f(x,y)=-\sqrt{x^2+y^2}$. What are the level curves? Sketch them for few values!

choose

$z=5 \leftarrow$ Graph $5=-\sqrt{x^2+y^2}$ (no solution)

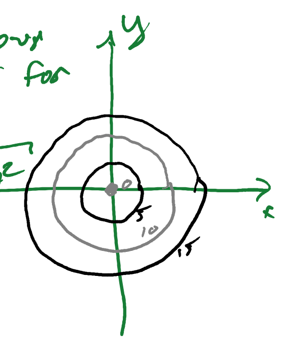
$z=0 \leftarrow$ Graph $0=\sqrt{x^2+y^2} \Rightarrow (x,y)=(0,0)$

$z=-5 \leftarrow -5=-\sqrt{x^2+y^2} \Rightarrow 25=x^2+y^2$
circle w/ radius 5.

$z=-10 \leftarrow -10=-\sqrt{x^2+y^2} \Rightarrow 100=x^2+y^2$

$z=-15 \leftarrow 225=x^2+y^2$
circle w/ radius 15
centered @ origin.

contour plot for
$f(x,y)=-\sqrt{x^2+y^2}$



❖ **Level Surfaces**

The concept of a level curve for a function of two variables can be extended to functions of three variables. If k is a constant, then an equation of the form $f(x,y,z)=k$ will, in general, represent a surface in 3D. The graph of this surface is called the level surface with constant k for the function f. They are generally more difficult to visualize than functions of two variables.

Example 5: Describe the level surface of the following functions.

a) $f(x,y,z)=x^2+y^2+z^2$

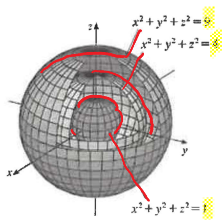
$k=1$: $1=x^2+y^2+z^2$
sphere. radius $=1$ centered @ origin.

$k=2$: $2=x^2+y^2+z^2$
sphere. radius $=\sqrt{2}$ centered @ origin

$k=3$: $3=x^2+y^2+z^2$
sphere w/ rad. $\sqrt{3}$

$k=4$: $4=x^2+y^2+z^2$

The level surfaces are spheres w/ radius $\sqrt{k}$ centered @ origin.



b) $f(x,y,z)=z^2-x^2-y^2$

$k=4$: $4=z^2-x^2-y^2$

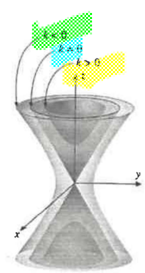
hyperboloid of 2 sheets.

see manipulate 15.17

$k=0$: $0=z^2-x^2-y^2$

cone

$k=-4$: $-4=z^2-x^2-y^2$
$z=0$: $-4=-x^2-y^2$
$4=x^2+y^2$
hyperboloid of 1 sheet.



Note: In the spherical example, the values of k pictured are not equally spaced.