13.4: Motion in Space

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Motion in Space

In this section we show the ideas of tangent and normal vectors and curvature can be used in physics to study the motion of an object, including its velocity and acceleration, along a space curve.

As we saw when derivatives were introduced, if $\vec{r}(t)$ gives the position of a particle at time t, then the **velocity vector** of the particle at time t is $\vec{v}(t) = \vec{r}\,'(t)$. Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

The **speed** of the particle at time t is the magnitude of the velocity vector, that is $|\vec{v}(t)|$. Thus we have:

$$v(t) = |\vec{v}(t)| = |\vec{r}\,'(t)| = \frac{ds}{dt}$$ ← notice the difference between v and $\vec{v}$

Similarly, the **acceleration vector** is $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$.

__Example 1__: Find the velocity, acceleration and speed of a particle with position vector $\vec{r}(t) = t^2\vec{i} + 4t\,\vec{j} + 4\ln t\,\vec{k}$.

velocity: $\vec{v}(t) = \vec{r}\,'(t) = \langle 2t, 4, \frac{4}{t} \rangle$

speed: $v = \sqrt{(2t^2) + 4^2 + (\frac{4}{t})^2}$

$= \sqrt{4t^2 + 16 + \frac{16}{t^2}}$

$= \sqrt{\frac{4t^4 + 16t^2 + 16}{t^2}}$

$= \sqrt{\frac{(2t^2+4)^2}{t^2}}$

$= \frac{2t^2+4}{t}$ ← scalar

acceleration: $\vec{a}(t) = \vec{r}\,''(t) = \langle 2, 0, -\frac{4}{t^2} \rangle$.

Example 2: Find the position vector of a moving object that has:

$$\vec{a}(t) = 2\vec{i} + 12t\vec{j} \qquad \vec{v}(0) = 7\vec{i} \qquad \vec{r}(0) = 2\vec{i} + 9\vec{k}$$

① $\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 2, 12t, 0\rangle\,dt$

$= \langle 2t, 6t^2, 0\rangle + \vec{C}_1$

② find $\vec{C}_1$: solve $\vec{v}(0) = \langle 0,0,0\rangle + \vec{C}_1 = \langle 7,0,0\rangle$

And $\vec{v}(t) = \langle 2t+7, 6t^2, 0\rangle$

③ $\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 2t+7, 6t^2, 0\rangle\,dt$

$= \langle t^2+7t, 2t^3, 0\rangle + \vec{D}$

④ find $\vec{D}$: solve $\vec{r}(0) = \langle 0,0,0\rangle + \vec{D} = \langle 2,0,9\rangle$

Therefore $\vec{r}(t) = \langle t^2+7t+2, 2t^3, 9\rangle$

If the force that acts on a particle is known, then the acceleration can be found from **Newton's Second Law of Motion**. The vector version of this law states that if, at any time t, a force $\vec{F}(t)$ acts on an object of mass m producing an acceleration $\vec{a}(t)$, then $\vec{F}(t) = m\vec{a}(t)$.

Example 3: What force is required so that a particle of mass $100Kg$ has the position function $\vec{r}(t) = t^3\vec{i} + t^2\vec{j} + t^3\vec{k}$ where r is in meters and t is in seconds?

recall: $\vec{F} = m\vec{a}$

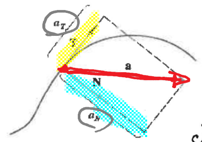
$\vec{v}(t) = \langle 3t^2, 2t, 3t^2\rangle$

and $\vec{a}(t) = \langle 6t, 2, 6t\rangle$

Thus $\vec{F}(t) = 100\langle 6t, 2, 6t\rangle$

❖ Tangent(ial) and Normal Components of Acceleration

When we study the motion of a particle, it is often useful to resolve the * acceleration into two components, one in the direction of the unit tangent vector and the other in the direction of the unit normal vector.



Derivation: Start: $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$\Rightarrow \vec{T} = \frac{\vec{v}}{v}$ ← velocity ← speed

* $\Rightarrow \vec{v} = v\vec{T}$

Differentiate both sides $\Rightarrow \frac{d}{dt}\vec{v} = \frac{d}{dt}v\vec{T}$

$\Rightarrow \vec{a} = v'\vec{T} + v\vec{T}'$

AND $T' = kv\vec{N}$

recall $\vec{N} = \frac{\vec{T}'}{|\vec{T}'|}$

$\Rightarrow \vec{T}' = |\vec{T}'|\vec{N}$

recall $k = \frac{|T'|}{|\vec{r}'|}$

$\Rightarrow |T'| = kv$

manipulate 14.34

Therefore $\boxed{\vec{a} = \underbrace{v'}_{a_T}\vec{T} + \underbrace{kv^2}_{a_N}\vec{N}}$

$\boxed{\begin{array}{l} a_T = v' \\ a_N = kv^2 \end{array}}$

Let's look at what this means. The first thing to notice is that the binormal vector is absent. No matter how an object moves through space, its acceleration always lies in the plane of $\vec{T}$ and $\vec{N}$ (the osculating plane). Next we notice that the tangential component of acceleration is v', the rate of change of speed, and the normal component of acceleration is κv^2, the curvature times the square of the speed. This makes sense if we think of a passenger in a car – a sharp turn in a road means a large value of the curvature κ, so the component of the acceleration perpendicular to the motion is large and the passenger is thrown against a car door.

As elsewhere, there are alternate versions of this formula, but the simple formula $\vec{a} = \underbrace{v'}_{a_T}\vec{T} + \underbrace{\kappa v^2}_{a_N}\vec{N}$ serves most of our purposes. Here a_T represents the tangential component of acceleration and a_N represents the normal component of acceleration.

Example 4: Decompose the acceleration vector of $\vec{r}(t) = \langle t^2, 2t, \ln t \rangle$ into its tangential and normal components.

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$
$$= v' \vec{T} + \kappa v^2 \vec{N}$$

And $a_T = \frac{\vec{r}' \cdot \vec{r}''}{|\vec{r}'|}$

And $a_N = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|}$

solution:

① $\vec{r}'(t) = \langle 2t, 2, \frac{1}{t} \rangle$

② $r''(t) = \langle 2, 0, -\frac{1}{t^2} \rangle$

③ $\vec{r}' \times \vec{r}'' = \langle -\frac{2}{t^2}, -(-\frac{\cancel{2t}}{t^2} - \frac{2}{t}), -4 \rangle = \langle -\frac{2}{t^2}, \frac{4}{t}, -4 \rangle$

$= 2 \langle -\frac{1}{t^2}, \frac{2}{t}, -2 \rangle$

④ $\vec{r}' \cdot \vec{r}'' = 4t + 0 - \frac{1}{t^3}$

$= \frac{4t^4 - 1}{t^3}$

⑤ $|\vec{r}'| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\frac{(2t^2+1)^2}{t^2}} = \frac{2t^2+1}{t}$

⑥ $|\vec{r}' \times \vec{r}''| = 2\sqrt{\frac{1}{t^4} + \frac{4}{t^2} + 4} = 2\sqrt{\frac{1 + 4t^2 + 4t^4}{t^4}} = 2\sqrt{\frac{(1+2t^2)^2}{t^4}} = \frac{2(1+2t^2)}{t^2}$

So $a_T = \frac{4t^4 - 1}{t^{\cancel{3}\,2}} \cdot \frac{\cancel{t}}{2t^2+1} = \frac{\cancel{(2t^2+1)}(2t^2-1)}{t^2\cancel{(2t^2+1)}} = \frac{2t^2-1}{t^2}$

And $a_N = \frac{2(1+2t^2)}{t^{\cancel{2}}} \cdot \frac{\cancel{t}}{2t^2+1} = \frac{2(1+2t^2)}{t(2t^2+1)}$

Thus $\vec{a} = \frac{2t^2-1}{t^2} \vec{T} + \frac{2(1+2t^2)}{t(2t^2+1)} \vec{N}$