12.6: Cylinders and Quadric Surfaces

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Cylinders and Quadric Surfaces

We have already looked at two special types of surfaces: planes and spheres. Here we investigate two other types of surfaces: cylinders and quadric surfaces.

In order to sketch the graph of a surface, it is useful to determine the curves of intersection of the surface with the planes parallel to the coordinate planes. These curves are called **traces** (or cross-sections) of the surface.

and

A **cylinder** is a surface that consists of all lines that are parallel to a given line a pass through a given plane curve.

The following are examples of cylinders:

manipulate about sketching
13.81

manipulate
13.76

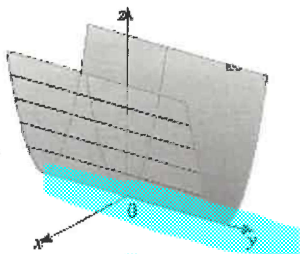


FIGURE 1
The surface $z = x^2$ is a parabolic cylinder.

No y

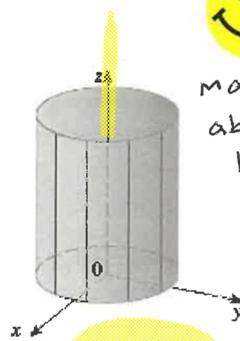


FIGURE 2 $x^2 + y^2 = 1$

No z

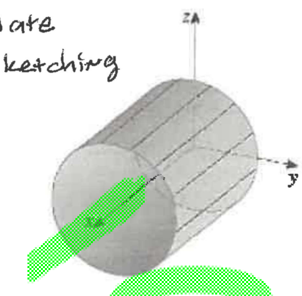


FIGURE 3 $y^2 + z^2 = 1$

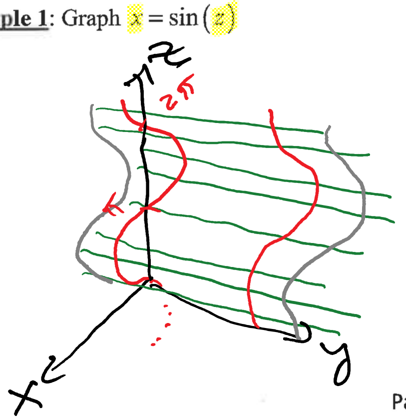
No x

Notice that in each equation a variable is missing. This is typical of a surface whose rulings are parallel to one of the coordinate axes. If one of the variables is missing from the equation of a surface, then the surface is a cylinder.

NOTE: When you are dealing with surfaces, it is important to recognize that an equation like $x^2 + y^2 = 1$ represents a cylinder and not a circle. The trace of the cylinder $x^2 + y^2 = 1$ in the *xy*-plane is the circle with the equations $x^2 + y^2 = 1, z = 0$.

Example 1: Graph $x = \sin(z)$

manipulate
13.82



A **quadric surface** is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

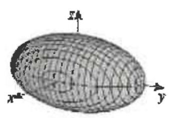
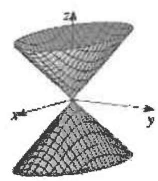
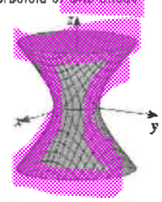
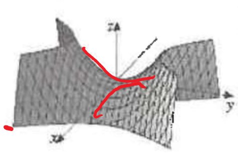
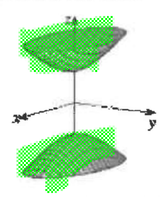
where $A, B, C, \ldots, J$ are constants, but by translation and rotation it can be brought into one of the two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

Quadric surfaces are the counterparts in three dimensions of the conic sections in the plane (parabola, ellipse, and hyperbola).

manipulates

13,83

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$. 13,87
13,84 Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative. 13,85
13,86 saddle. Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets. 13,88

http://demonstrations.wolfram.com/PlaneSectionsOfSurfaces/

NOTE: For Hyperboloid you can think of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\pm 1$. If we have $+1$ then we have hyperboloid of one sheet. Otherwise it is a hyperboloid of two sheets. Since the coefficient of z is negative the hyperboloids open toward z-axis.

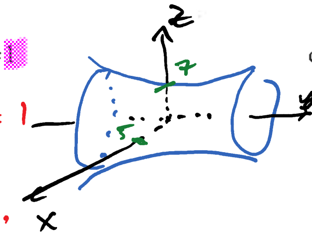
Example 2: Identify each quadric surface and graph the cone, and elliptic paraboloid if any!

a) $\frac{x^2}{25}-\frac{y^2}{25}+\frac{z^2}{49}=1$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

hyperboloid of 1 sheet.

Along y-axis

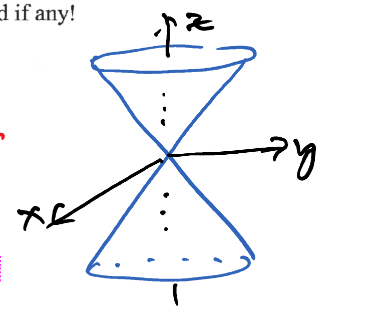


d) $+\frac{z^2}{25}-\frac{y^2}{9}-x^2=0$

$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

cone

Along z-axis

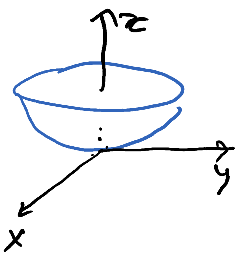


b) $3z=9x^2+y^2$

$\frac{z}{c}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}$

elliptic paraboloid.

along the z-axes

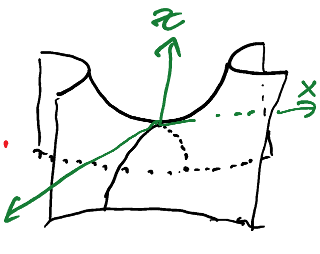


e) $z=\frac{x^2}{4}-\frac{y^2}{2}$

$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

hyperbolic paraboloid.

saddle you can sit in.

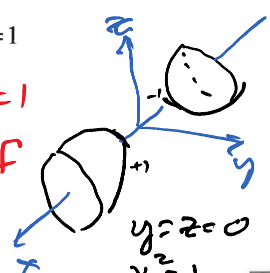


c) $-x^2-3y^2+9z^2=1$

$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$

hyperboloid of 2 sheets

Along x-axis.



$y=z=0$
$x^2=1$

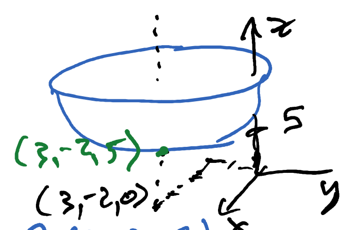
f) $(z-5)=(x-3)^2+(y+2)^2$

$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

elliptic paraboloid.

vertical

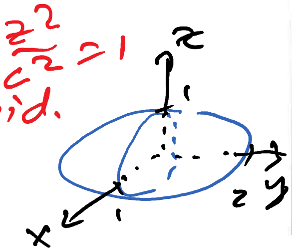
center @ $(3,-2,5)$



g) $x^2+\left(\frac{y}{4}\right)^2+z^2=1$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

ellipsoid.



Math is crazy: Two spheres can be formed from one! Yup, that's right.

The Banach–Tarski paradox states: Given a solid ball in three-dimensional space, there exists a decomposition of the ball into a finite number of disjoint subsets, which can then be put back together in a different way to yield two identical copies of the original ball. Indeed, the reassembly process involves only moving the pieces around and rotating them without changing their shape