10.5: Conic Sections

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Conic Sections

Conic sections are the curves we get when we make a straight cut in a cone. As shown below, there are four interesting conic sections: circles, ellipses, parabolas and hyperbolas.

Circle Ellipse Parabola Hyperbola

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Note: The circle is a special case of the ellipse.

❖ **General Equation of a Conic Section**

The graph of the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Where A and C are not both 0, is a conic.

- If A or C is 0, it's a parabola.
- If A and C are equal, it's a circle.
- If A and C have the same sign, it's an ellipse.
- If A and C have opposite signs, it's a hyperbola.

Example 1: Identify the type of conic section whose equation is given.

a) $x^2 = y^2 + 1$ $\Rightarrow$ $x^2 - y^2 - 1 = 0$ hyperbola

b) $y^2 - 8y = 6x - 16$ $\Rightarrow$ $y^2 - 8y - 6x + 16 = 0$ parabola

c) $4x^2 + 4x + y^2 = 10$ $\Rightarrow$ $4x^2 + 4x + y^2 - 10 = 0$ ellipse or circle

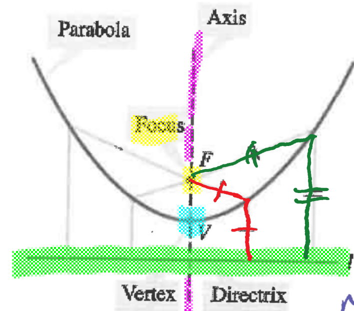
Historical note: The history of conic sections begins around 300 BC and continued for nearly 2,000 years. For most of this time, conics lived entirely within the realm of geometry. It was about diagrams, straight edges, angles, and constructions. But in the 1600's, Descartes and Fermat found a way to represent geometry algebraically. A taste of this **analytic geometry** is in this section.

❖ Parabolas, a Geometric Definition

A **parabola** is the set of points in the plane equidistant from a fixed point F (called the focus) and a fixed line l (called the directrix).

The **vertex** of the parabola is half way between the focus and the directrix and **axis of symmetry is** the line that runs through the focus perpendicular to the directrix.

Note: Pay attention to explanations that are from geometry vs. analytic geometry.



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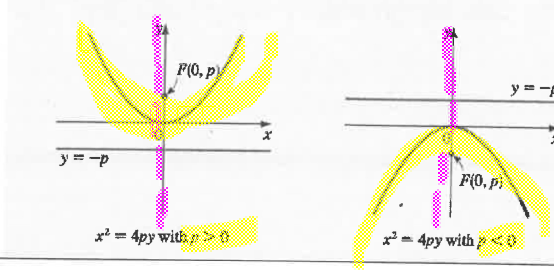
Parabola with vertical axis: The graph of the equation $x^2 = 4py$ is a parabola with the following properties:

Vertex: $V(0,0)$

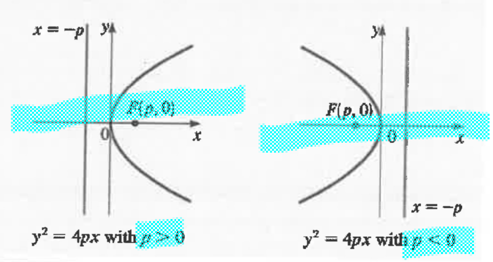
Focus: $F(0,p)$

Directrix: $y=-p$

The parabola opens upward if $p > 0$ or downward if $p < 0$.



The parabola opens to the right if $p > 0$ or to the left if $p < 0$.



Parabola with horizontal axis: The graph of the equation $y^2 = 4px$ is a parabola with the following properties:

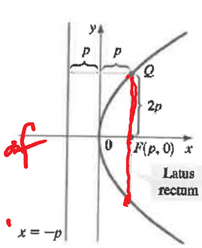
Vertex: $V(0,0)$

Focus: $F(p,0)$

Directrix: $x=-p$

We can use the coordinates of the focus to estimate the *width* of a parabola when sketching the graph. The line segment that runs through the focus perpendicular to the axis, with end points on the parabola, is called the **latus rectum**, and its length is the focal **diameter** of the parabola. This length is $|4p|$.

reminds us of focal length.

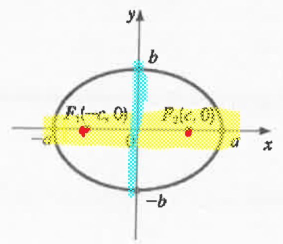
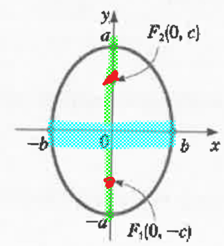


❖ Ellipses, a Geometric Definition

An **ellipse** is the set of all points in the plane that the sum of their distances from two fixed points called the **foci** is constant.

ELLIPSE WITH CENTER AT THE ORIGIN

The graph of each of the following equations is an ellipse with center at the origin and having the given properties.

EQUATION	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $a>b>0$	$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ $a>b>0$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
MAJOR AXIS	Horizontal, length $2a$	Vertical, length $2a$
MINOR AXIS	Vertical, length $2b$	Horizontal, length $2b$
FOCI	$(\pm c, 0)$, $c^2 = a^2 - b^2$	$(0, \pm c)$, $c^2 = a^2 - b^2$
GRAPH		

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In the standard equation for an ellipse, a^2 is the larger denominator and b^2 is the smaller. To find c^2, subtract the smaller denominator from the larger one.

Example 2: Find the vertices and foci of the ellipse $x^2+3y^2+2x-12y+\cancel{10}^{-20}=0$ and sketch its graph.

$\Rightarrow x^2+2x+1 + 3y^2-12y \quad \cancel{+10}^{-20} = 0+1$

$\Rightarrow (x+1)^2 + 3(y^2-4y+4) \cancel{+10}^{-20} = 1+4$

$\Rightarrow \frac{1(x-(-1))^2}{25} + \frac{(y-2)^2}{25/3} = \frac{25}{25}$

ellipse centered @ $(-1, 2)$

$x=-1$ $\quad$ $y=2$

$3(y-2)^2=25$

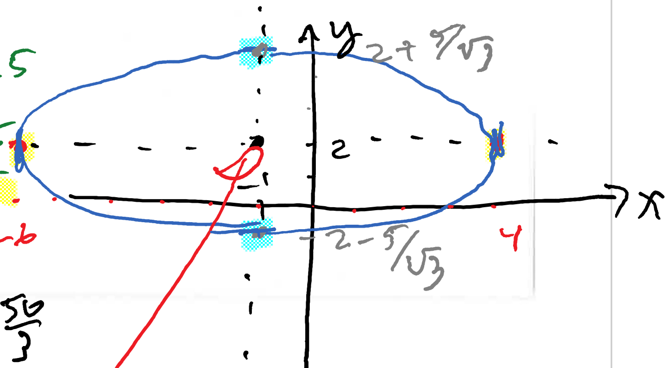
$y-2=\pm\sqrt{\frac{25}{3}}$

$y=2\pm\frac{5}{\sqrt{3}}$

$(x+1)^2=25$

$x+1=\pm 5$

$x=-1\pm 5$



Foci: $(\pm c, 0)$, $c^2=a^2-b^2=\frac{50}{3}$

$\left(\pm\sqrt{\frac{50}{3}}, 0\right)$ $\quad 25 \quad \frac{25}{3}$

$\left(\frac{\pm\sqrt{50}}{2}, 0\right)$ 25 $\frac{25}{3}$

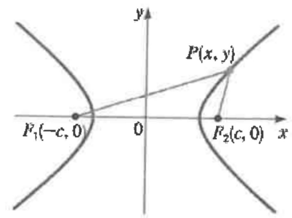
Accounting for the shift, the foci: $\left((-1) \pm \frac{\sqrt{50}}{3}, (2)\right)$

❖ Hyperbolas, a Geometric Definition

A **hyperbola** is the set of all points in the plane that the difference of their distances from two fixed points called the **foci** is constant.

The segment joining the two vertices is the **transverse axis** of the hyperbola.

The **asymptotes** are lines that the hyperbola approaches for large values of x and y.



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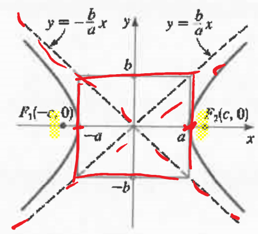
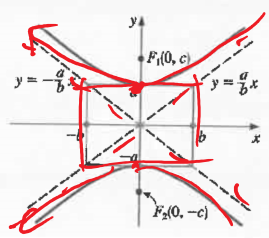
HYPERBOLA WITH CENTER AT THE ORIGIN

The graph of each of the following equations is a hyperbola with center at the origin and having the given properties.

EQUATION	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad (a > 0, b > 0)$
VERTICES	$(\pm a, 0)$	$(0, \pm a)$
TRANSVERSE AXIS	Horizontal, length $2a$	Vertical, length $2a$
ASYMPTOTES	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$
FOCI	$(\pm c, 0), \quad c^2 = a^2 + b^2$	$(0, \pm c), \quad c^2 = a^2 + b^2$
GRAPH		

How to Sketch a Hyperbola centered at the origin

1. Sketch the central box. This is the rectangle centered at the origin, with sides parallel to the axes, that crosses on axis at $\pm a$, the other at $\pm b$.
2. Sketch the Asymptotes. These are the lines obtained by extending the diagonals of the central box.
3. Plot the Vertices. These are the two x-intercepts or the two y-intercepts.
4. Sketch the Hyperbola. Start at a vertex and sketch a branch of the hyperbola, approaching the asymptotes. Sketch the other branch in the same way.