11 Review

Concept Check

1. (a) What is a convergent sequence?
   (b) What is a convergent series?
   (c) What does $\lim_{n\to\infty} a_n = 3$ mean?
   (d) What does $\sum_{n=1}^{\infty} a_n = 3$ mean?

2. (a) What is a bounded sequence?
   (b) What is a monotonic sequence?
   (c) What can you say about a bounded monotonic sequence?

3. (a) What is a geometric series? Under what circumstances is it convergent? What is its sum?
   (b) What is a p-series? Under what circumstances is it convergent?

4. Suppose $\sum a_n = 3$ and s_n is the nth partial sum of the series. What is $\lim_{n\to\infty} a_n$? What is $\lim_{n\to\infty} s_n$?

5. State the following.
   (a) The Test for Divergence
   (b) The Integral Test
   (c) The Comparison Test
   (d) The Limit Comparison Test
   (e) The Alternating Series Test
   (f) The Ratio Test
   (g) The Root Test

6. (a) What is an absolutely convergent series?
   (b) What can you say about such a series?
   (c) What is a conditionally convergent series?

7. (a) If a series is convergent by the Integral Test, how do you estimate its sum?
   (b) If a series is convergent by the Comparison Test, how do you estimate its sum?
   (c) If a series is convergent by the Alternating Series Test, how do you estimate its sum?

8. (a) Write the general form of a power series.
   (b) What is the radius of convergence of a power series?
   (c) What is the interval of convergence of a power series?

9. Suppose $f(x)$ is the sum of a power series with radius of convergence R.
   (a) How do you differentiate f? What is the radius of convergence of the series for f'?
   (b) How do you integrate f? What is the radius of convergence of the series for $\int f(x)\,dx$?

10. (a) Write an expression for the nth-degree Taylor polynomial of f centered at a.
    (b) Write an expression for the Taylor series of f centered at a.
    (c) Write an expression for the Maclaurin series of f.
    (d) How do you show that $f(x)$ is equal to the sum of its Taylor series?
    (e) State Taylor's Inequality.

11. Write the Maclaurin series and the interval of convergence for each of the following functions.
    (a) $1/(1-x)$ (b) e^x
    (c) $\sin x$ (d) $\cos x$
    (e) $\tan^{-1}x$ (f) $\ln(1+x)$

12. Write the binomial series expansion of $(1+x)^k$. What is the radius of convergence of this series?

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ is convergent.
2. The series $\sum_{n=1}^{\infty} n^{-\sin 1}$ is convergent.
3. If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} a_{2n+1} = L$.
4. If $\sum c_n 6^n$ is convergent, then $\sum c_n(-2)^n$ is convergent.
5. If $\sum c_n 6^n$ is convergent, then $\sum c_n(-6)^n$ is convergent.
6. If $\sum c_n x^n$ diverges when $x = 6$, then it diverges when $x = 10$.
7. The Ratio Test can be used to determine whether $\sum 1/n^3$ converges.
8. The Ratio Test can be used to determine whether $\sum 1/n!$ converges.
9. If $0 \leqslant a_n \leqslant b_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$
11. If $-1 < \alpha < 1$, then $\lim_{n\to\infty} \alpha^n = 0$.
12. If $\sum a_n$ is divergent, then $\sum |a_n|$ is divergent.
13. If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \cdots$ converges for all x, then $f'''(0) = 2$.
14. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n + b_n\}$ is divergent.
15. If $\{a_n\}$ and $\{b_n\}$ are divergent, then $\{a_n b_n\}$ is divergent.
16. If $\{a_n\}$ is decreasing and $a_n > 0$ for all n, then $\{a_n\}$ is convergent.
17. If $a_n > 0$ and $\sum a_n$ converges, then $\sum (-1)^n a_n$ converges.

18. If $a_n > 0$ and $\lim_{n\to\infty} (a_{n+1}/a_n) < 1$, then $\lim_{n\to\infty} a_n = 0$.

19. $0.99999\ldots = 1$

20. If $\lim\limits_{n\to\infty} a_n = 2$, then $\lim\limits_{n\to\infty} (a_{n+3} - a_n) = 0$.

21. If a finite number of terms are added to a convergent series, then the new series is still convergent.

22. If $\sum\limits_{n=1}^{\infty} a_n = A$ and $\sum\limits_{n=1}^{\infty} b_n = B$, then $\sum\limits_{n=1}^{\infty} a_n b_n = AB$.

Exercises

1–8 Determine whether the sequence is convergent or divergent. If it is convergent, find its limit.

1. $a_n = \dfrac{2 + n^3}{1 + 2n^3}$

2. $a_n = \dfrac{9^{n+1}}{10^n}$

3. $a_n = \dfrac{n^3}{1 + n^2}$

4. $a_n = \cos(n\pi/2)$

5. $a_n = \dfrac{n \sin n}{n^2 + 1}$

6. $a_n = \dfrac{\ln n}{\sqrt{n}}$

7. $\{(1 + 3/n)^{4n}\}$

8. $\{(-10)^n/n!\}$

9. A sequence is defined recursively by the equations $a_1 = 1$, $a_{n+1} = \frac{1}{3}(a_n + 4)$. Show that $\{a_n\}$ is increasing and $a_n < 2$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

10. Show that $\lim_{n\to\infty} n^4 e^{-n} = 0$ and use a graph to find the smallest value of N that corresponds to $\varepsilon = 0.1$ in the precise definition of a limit.

11–22 Determine whether the series is convergent or divergent.

11. $\sum\limits_{n=1}^{\infty} \dfrac{n}{n^3 + 1}$

12. $\sum\limits_{n=1}^{\infty} \dfrac{n^2 + 1}{n^3 + 1}$

13. $\sum\limits_{n=1}^{\infty} \dfrac{n^3}{5^n}$

14. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n + 1}}$

15. $\sum\limits_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$

16. $\sum\limits_{n=1}^{\infty} \ln\left(\dfrac{n}{3n + 1}\right)$

17. $\sum\limits_{n=1}^{\infty} \dfrac{\cos 3n}{1 + (1.2)^n}$

18. $\sum\limits_{n=1}^{\infty} \dfrac{n^{2n}}{(1 + 2n^2)^n}$

19. $\sum\limits_{n=1}^{\infty} \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{5^n n!}$

20. $\sum\limits_{n=1}^{\infty} \dfrac{(-5)^{2n}}{n^2 9^n}$

21. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} \dfrac{\sqrt{n}}{n + 1}$

22. $\sum\limits_{n=1}^{\infty} \dfrac{\sqrt{n + 1} - \sqrt{n - 1}}{n}$

23–26 Determine whether the series is conditionally convergent, absolutely convergent, or divergent.

23. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} n^{-1/3}$

24. $\sum\limits_{n=1}^{\infty} (-1)^{n-1} n^{-3}$

25. $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n (n + 1)3^n}{2^{2n+1}}$

26. $\sum\limits_{n=2}^{\infty} \dfrac{(-1)^n \sqrt{n}}{\ln n}$

27–31 Find the sum of the series.

27. $\sum\limits_{n=1}^{\infty} \dfrac{(-3)^{n-1}}{2^{3n}}$

28. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n + 3)}$

29. $\sum\limits_{n=1}^{\infty} [\tan^{-1}(n + 1) - \tan^{-1} n]$

30. $\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n \pi^n}{3^{2n}(2n)!}$

31. $1 - e + \dfrac{e^2}{2!} - \dfrac{e^3}{3!} + \dfrac{e^4}{4!} - \cdots$

32. Express the repeating decimal $4.17326326326\ldots$ as a fraction.

33. Show that $\cosh x \geqslant 1 + \frac{1}{2}x^2$ for all x.

34. For what values of x does the series $\sum_{n=1}^{\infty} (\ln x)^n$ converge?

35. Find the sum of the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n^5}$ correct to four decimal places.

36. (a) Find the partial sum s_5 of the series $\sum_{n=1}^{\infty} 1/n^6$ and estimate the error in using it as an approximation to the sum of the series.
(b) Find the sum of this series correct to five decimal places.

37. Use the sum of the first eight terms to approximate the sum of the series $\sum_{n=1}^{\infty} (2 + 5^n)^{-1}$. Estimate the error involved in this approximation.

38. (a) Show that the series $\sum\limits_{n=1}^{\infty} \dfrac{n^n}{(2n)!}$ is convergent.
(b) Deduce that $\lim\limits_{n\to\infty} \dfrac{n^n}{(2n)!} = 0$.

39. Prove that if the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the series

$$\sum_{n=1}^{\infty} \left(\frac{n + 1}{n}\right) a_n$$

is also absolutely convergent.

40–43 Find the radius of convergence and interval of convergence of the series.

40. $\sum\limits_{n=1}^{\infty} (-1)^n \dfrac{x^n}{n^2 5^n}$

41. $\sum\limits_{n=1}^{\infty} \dfrac{(x + 2)^n}{n 4^n}$

Graphing calculator or computer required

42. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n(x-2)^n}{(n+2)!}$

43. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n(x-3)^n}{\sqrt{n+3}}$

44. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$$

45. Find the Taylor series of $f(x) = \sin x$ at $a = \pi/6$.

46. Find the Taylor series of $f(x) = \cos x$ at $a = \pi/3$.

47–54 Find the Maclaurin series for f and its radius of convergence. You may use either the direct method (definition of a Maclaurin series) or known series such as geometric series, binomial series, or the Maclaurin series for e^x, $\sin x$, $\tan^{-1}x$, and $\ln(1 + x)$.

47. $f(x) = \dfrac{x^2}{1+x}$

48. $f(x) = \tan^{-1}(x^2)$

49. $f(x) = \ln(4 - x)$

50. $f(x) = xe^{2x}$

51. $f(x) = \sin(x^4)$

52. $f(x) = 10^x$

53. $f(x) = 1/\sqrt[4]{16 - x}$

54. $f(x) = (1 - 3x)^{-5}$

55. Evaluate $\displaystyle\int \frac{e^x}{x}\,dx$ as an infinite series.

56. Use series to approximate $\int_0^1 \sqrt{1 + x^4}\,dx$ correct to two decimal places.

57–58
(a) Approximate f by a Taylor polynomial with degree n at the number a.
(b) Graph f and T_n on a common screen.
(c) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the given interval.
(d) Check your result in part (c) by graphing $|R_n(x)|$.

57. $f(x) = \sqrt{x}$, $a = 1$, $n = 3$, $0.9 \leqslant x \leqslant 1.1$

58. $f(x) = \sec x$, $a = 0$, $n = 2$, $0 \leqslant x \leqslant \pi/6$

59. Use series to evaluate the following limit.

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

60. The force due to gravity on an object with mass m at a height h above the surface of the earth is

$$F = \frac{mgR^2}{(R+h)^2}$$

where R is the radius of the earth and g is the acceleration due to gravity for an object on the surface of the earth.
(a) Express F as a series in powers of h/R.
(b) Observe that if we approximate F by the first term in the series, we get the expression $F \approx mg$ that is usually used when h is much smaller than R. Use the Alternating Series Estimation Theorem to estimate the range of values of h for which the approximation $F \approx mg$ is accurate to within one percent. (Use $R = 6400$ km.)

61. Suppose that $f(x) = \sum_{n=0}^{\infty} c_n x^n$ for all x.
(a) If f is an odd function, show that

$$c_0 = c_2 = c_4 = \cdots = 0$$

(b) If f is an even function, show that

$$c_1 = c_3 = c_5 = \cdots = 0$$

62. If $f(x) = e^{x^2}$, show that $f^{(2n)}(0) = \dfrac{(2n)!}{n!}$.

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True-False Quiz

1. False **3.** True **5.** False **7.** False **9.** False
11. True **13.** True **15.** False **17.** True
19. True **21.** True

Exercises

1. $\frac{1}{2}$ **3.** D **5.** 0 **7.** e^{12} **9.** 2 **11.** C **13.** C
15. D **17.** C **19.** C **21.** C **23.** CC **25.** AC
27. $\frac{1}{11}$ **29.** $\pi/4$ **31.** e^{-e} **35.** 0.9721
37. 0.18976224, error $< 6.4 \times 10^{-7}$
41. 4, $[-6, 2)$ **43.** 0.5, $[2.5, 3.5)$

45. $\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n}+\frac{\sqrt{3}}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}\right]$

47. $\sum_{n=0}^{\infty}(-1)^n x^{n+2}$, $R = 1$ **49.** $\ln 4 - \sum_{n=1}^{\infty}\frac{x^n}{n4^n}$, $R = 4$

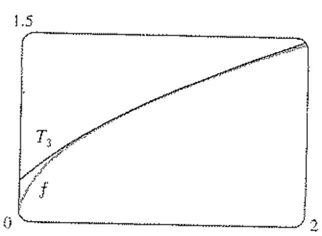
51. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{8n+4}}{(2n+1)!}$, $R = \infty$

53. $\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1\cdot 5\cdot 9\cdot \cdots \cdot (4n-3)}{n!2^{6n+1}}x^n$, $R = 16$

55. $C + \ln|x| + \sum_{n=1}^{\infty}\frac{x^n}{n\cdot n!}$

57. (a) $1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3$

(b)



(c) 0.000006

59. $-\frac{1}{6}$

PROBLEMS PLUS ■ PAGE 805

1. $15!/5! = 10{,}897{,}286{,}400$
3. (b) 0 if $x = 0$, $(1/x) - \cot x$ if $x \neq k\pi$, k an integer
5. (a) $s_n = 3 \cdot 4^n$, $l_n = 1/3^n$, $p_n = 4^n/3^{n-1}$ (c) $\frac{2}{5}\sqrt{3}$

9. $(-1, 1)$, $\frac{x^3+4x^2+x}{(1-x)^4}$

11. $\ln\frac{1}{2}$ **13.** (a) $\frac{250}{101}\pi(e^{-(n-1)\pi/5}-e^{-n\pi/5})$ (b) $\frac{250}{101}\pi$

19. $\frac{\pi}{2\sqrt{3}} - 1$

21. $-\left(\frac{\pi}{2}-\pi k\right)^2$ where k is a positive integer