Name: key
Assessment 7
Math 220: Linear Algebra

Instructions: Please carefully complete these questions by hand. Be sure to show all work (this includes notating steps in row reduction in matrices that include a variable).

Should you choose to work these on scratch paper, **please do not put more than one question on a page.** Additional sheets of paper are acceptable. Write your name on every page. **You can submit more pages, but Gradescope will not accept less pages than the original assignment.**

Upload your solutions to Gradescope by 8 am on Monday (3/15). During your presentation time, you will be asked to explain your thought process and reasoning on a randomly assigned question. Late submissions (or resubmissions) are available thru 5 pm with a 5% penalty. Resubmission is helpful if you think you can gain 5% in the process.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1TE17S-z6oWrdejMbX5txwWX-JowaTFspdJ-7pCtpodY/edit?usp=sharing

Reminders: It is okay to collaborate with peers and use online resources. However, the final work should be your own and you should be prepared to present on each question.

(1.1) For the matrix $\begin{bmatrix} 9 & -2 & 8 & -7 \\ 0 & 7 & h & 0 \\ 0 & 0 & 9 & 5 \\ 0 & 0 & 0 & 3 \end{bmatrix}$, find values of h such that the dimension of the eigenspace corresponding to $\lambda = 9$ is one. Find h such that the dimension is two.

$$A-9I = \begin{bmatrix} 0 & -2 & 8 & -7 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 0 & -2 & 8 & 0 \\ 0 & -2 & h & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $h = 8$

$$A-9I \sim \begin{bmatrix} 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And the eigenspace corresponding to $\lambda = 9$ has dimension 2

If $h \neq 8$

$$A-9I \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And the eigenspace corresponding to $\lambda = 9$ has dimension 1.

(1.2) Find (and simplify) the characteristic polynomial of $\begin{bmatrix} 7 & 8 & 3 \\ 0 & 2 & 0 \\ 7 & 6 & -6 \end{bmatrix}$

$$\begin{vmatrix} 7-\lambda & 8 & 3 \\ 0 & 2-\lambda & 0 \\ 7 & 6 & -6-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 7-\lambda & 3 \\ 7 & -6-\lambda \end{vmatrix}$$

$$= (2-\lambda)\left[(7-\lambda)(-6-\lambda)-21\right]$$

$$-42-\lambda+\lambda^2-21$$

$$= (2-\lambda)(\lambda^2-\lambda-63)$$

$$= -\lambda^3+3\lambda^2+61\lambda-126$$

(1.3) Find the eigenvalues (and their multiplicity) of $\begin{bmatrix} 9 & 7 \\ -5 & 5 \end{bmatrix}$

solve $\begin{vmatrix} 9-\lambda & 7 \\ -5 & 5-\lambda \end{vmatrix} = 0$

$\Rightarrow (9-\lambda)(5-\lambda) + 35 = 0$

$\Rightarrow 45 - 14\lambda + \lambda^2 + 35 = 0$

$\Rightarrow \lambda^2 - 14\lambda + 80 = 0$

$$\Rightarrow \lambda = \frac{14 \pm \sqrt{196 - 4(1)(80)}}{2(1)}$$

$$= \frac{14 \pm \sqrt{-124}}{2}$$

$$= 7 \pm i\sqrt{31}$$

(1.4) Diagonalize $A = \begin{bmatrix} 8 & -4 & 0 & 8 \\ 0 & 5 & 1 & -7 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

$\lambda = 4$:

$$\text{rref}(A - 4I) = \begin{bmatrix} 1 & 0 & 1 & -5 \\ 0 & 1 & 1 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

eigenvecs $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 5$:

$$\text{rref}(A - 5I) = \begin{bmatrix} 1 & -4/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

eigenvec $\begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix}$

$\lambda = 8$

$$\text{rref}(A - 8I) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

eigen vec $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

so $A = PDP^{-1}$

w/ $D = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$ and $P = \begin{bmatrix} -1 & 5 & 4 & 1 \\ -1 & 7 & 3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(1.5) Diagonalize $\begin{bmatrix} -10 & -18 & 9 \\ 12 & 20 & -9 \\ 12 & 18 & -7 \end{bmatrix}$

solve $0 = \begin{vmatrix} -10-\lambda & -18 & 9 \\ 12 & 20-\lambda & -9 \\ 12 & 18 & -7-\lambda \end{vmatrix}$

$= (-10-\lambda)\begin{vmatrix} 20-\lambda & -9 \\ 18 & -7-\lambda \end{vmatrix} + 18\begin{vmatrix} 12 & -9 \\ 12 & -7-\lambda \end{vmatrix} + 9\begin{vmatrix} 12 & 20-\lambda \\ 12 & 18 \end{vmatrix}$

$= (-10-\lambda)\left[\underbrace{(20-\lambda)(-7-\lambda)+162}_{\lambda^2-13\lambda+22=(\lambda-11)(\lambda-2)}\right] + 18\left[\underbrace{12(-7-\lambda)+108}_{24-12\lambda=-12(\lambda-2)}\right]$
$+ 9\left[\underbrace{216-12(20-\lambda)}_{-24+12\lambda=12(\lambda-2)}\right]$

$= (-10-\lambda)(\lambda-11)(\lambda-2) - 216(\lambda-2) - 108(\lambda-2)$

$= (\lambda-2)\left[-\lambda^2+\lambda+110-216+108\right]$

$= -(\lambda-2)\left[\lambda^2-\lambda+2\right] = -(\lambda-2)(\lambda-2)(\lambda+1)$

eigenvals are $\lambda = 2$ (mult 2) and $\lambda = -1$

$\lambda = 2$

eigenvectors

$A - 2I \sim \begin{bmatrix} 1 & \frac{3}{2} & -\frac{3}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$

$A + I \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

conclusion

so $A = PDP^{-1}$

where

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

AND

$P = \begin{bmatrix} -3 & 3 & -1 \\ 2 & 0 & 1 \\ 0 & 4 & 1 \end{bmatrix}$

(1.6) Prove that if $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

claim: If $n \times n$ matrices A & B are similar then they have the same characteristic polynomial and eigenvalues.

proof.

Let similar matrices A & B be given.

$\Rightarrow \exists$ invertible P s.t. $B = P^{-1} A P$

$$\Rightarrow B - \lambda I = P^{-1} A P - \lambda I$$
$$= P^{-1} A P - P^{-1} \lambda I P$$
$$= P^{-1} (A - \lambda I) P$$

$$\Rightarrow \det(B - \lambda I) = \det(P^{-1}(A - \lambda I)P)$$
$$= \det(P^{-1}) \det(A - \lambda I) \det(P)$$
$$= \det(A - \lambda I)$$

$\therefore$ A & B have the same characteristic polynomial and eigenvalues.

(1.7) If $A = \begin{bmatrix} 2 & -3/2 \\ 1 & -1/2 \end{bmatrix}$, calculate A^{1000} without the aid of a calculator.

solve $\begin{vmatrix} 2-\lambda & -3/2 \\ 1 & -\frac{1}{2}-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)(-\frac{1}{2}-\lambda) + \frac{3}{2} = 0$

$\Rightarrow \frac{1}{2} - \frac{3}{2}\lambda + \lambda^2 = 0$

$\Rightarrow 2\lambda^2 - 3\lambda + 1 = 0$

$\Rightarrow (2\lambda - 1)(\lambda - 1) = 0$

$\Rightarrow \lambda = \frac{1}{2}$ or $\lambda = 1$

$\lambda = 1$: $A - I = \begin{bmatrix} 1 & -3/2 \\ 1 & -3/2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$ eigenvec $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda = \frac{1}{2}$: $A - \frac{1}{2}I = \begin{bmatrix} 3/2 & -3/2 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ eigenvec $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A = PDP^{-1}$ w/ $P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$

$\Rightarrow A^{1000} = PD^{1000}P^{-1} \approx P \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$ w/ $P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$

$PD^{1000} = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}$

$PD^{1000}P^{-1} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$

(1.8) Explain why/where diagonalization is helpful.

Diagonalization is helpful when you need to find powers of a matrix because $A^n = P D^n P^{-1}$ and D^n is easy to find.

We need powers of matrices when we are working w/ iterative processes like Markov chains.