Name: Key

Assessment 6

Math 220: Linear Algebra

Instructions: Please carefully complete these questions by hand. Be sure to show all work (this includes notating steps in row reduction in matrices that include a variable).

Should you choose to work these on scratch paper, **please do not put more than one question on a page.** Additional sheets of paper are acceptable. Write your name on every page. **You can submit more pages, but Gradescope will not accept less pages than the original assignment.**

Upload your solutions to Gradescope by 8 am on Monday (3/8). During your presentation time, you will be asked to explain your thought process and reasoning on a randomly assigned question. Late submissions (or resubmissions) are available thru 5 pm with a 5% penalty. Resubmission is helpful if you think you can gain 5% in the process.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1TE17S-z6oWrdejMbX5txwWX-JowaTFspdJ-7pCtpodY/edit?usp=sharing

Reminders: It is okay to collaborate with peers and use online resources. However, the final work should be your own and you should be prepared to present on each question.

(1.1) If the nullity of a 4 x 6 matrix A is 3, what are the dimensions of the column and row spaces of A?

$$\underbrace{\text{rank}}_{3} + \underbrace{\text{nullity}}_{3} = \#\text{ of columns}$$

dim(col) = 3

dim(row) = 3

(1.2) Consider $A = \begin{bmatrix} 1 & 0 & 7 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Determine a basis for, and find the dimensions of Nul A, Col A, and Row A.

basis for null(A): $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ w/ nullity = 1

basis for col(A): $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix} \right\}$ w/ rank = 3

basis for row(A): $\{ (1, 0, 7, 3), (0, 0, 1, -3), (0, 0, 0, 1) \}$
w/ dim(row) = 3.

claim:

(1.3) If $\vec{v}_1, \ldots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an n x n matrix A, prove that the set $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly independent.

proof.

Suppose $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is L.D.

Since $\vec{v}_1 \neq \vec{0}$ at least one of the vectors is a lin. comb of the preceding. Let's call the first such vector $\vec{v}_{p+1}$,

$\Rightarrow \exists$ scalars $c_1, \ldots, c_p$ s.t. (*)

$$c_1 \vec{v}_1 + \ldots + c_p \vec{v}_p = \vec{v}_{p+1}$$

$$\Rightarrow c_1 A\vec{v}_1 + \ldots + c_p A\vec{v}_p = A\vec{v}_{p+1}$$

$$\Rightarrow c_1 \lambda_1 \vec{v}_1 + \ldots + c_p \lambda_p \vec{v}_p = \lambda_{p+1} \vec{v}_{p+1}$$

And multiplying both sides of (*) by λ_{p+1}

$$c_1 \lambda_{p+1} \vec{v}_1 + \ldots + c_p \lambda_{p+1} \vec{v}_p = \lambda_{p+1} \vec{v}_{p+1}$$

subtracting

$$c_1(\lambda_1 - \lambda_{p+1})\vec{v}_1 + \ldots + c_p(\lambda_p - \lambda_{p+1})\vec{v}_p = \vec{0}$$

since $\vec{v}_1, \ldots, \vec{v}_p$ are LI, the weights are all non-zero. But λ's are distinct so the c's are also zero and hence $\vec{v}_{p+1} = \vec{0}$

$\Rightarrow\Leftarrow$

$\therefore \{\vec{v}_1, \ldots, \vec{v}_r\}$ are L.I.

(1.4) In P_2, find the change-of-coordinates matrix from $B = \{1+2t^2, 3+t+7t^2, 1-3t\}$ to the standard basis. Then write t^2 as a linear combination of the polynomials in B. Please clearly describe any mappings to/from P_2.

$$a + bt + ct^2 \iff \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$1 + 2t^2 \iff \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$3 + t + 7t^2 \iff \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$$

$$1 - 3t \iff \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$$

$$P_B = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & -3 \\ 2 & 7 & 0 \end{bmatrix}$$

$$P_B^{-1} = \begin{bmatrix} 21 & 7 & -10 \\ -6 & -2 & 3 \\ -2 & -1 & 1 \end{bmatrix}$$

$$t^2 \iff \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad P_B^{-1}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}_B$$

which is to say $t^2 = -10(1+2t^2) + 3(3+t+7t^2) + 1(1-3t)$

(1.5) Let $B=\left\{\begin{bmatrix}-1\\4\end{bmatrix},\begin{bmatrix}1\\-3\end{bmatrix}\right\}$ and $C=\left\{\begin{bmatrix}1\\4\end{bmatrix},\begin{bmatrix}1\\3\end{bmatrix}\right\}$ be bases for $\mathbb{R}^2$. Find the change-of-coordinates matrix from C to B. Draw the diagram that relates $\vec{x},[\vec{x}]_B,[\vec{x}]_C$.

$$\begin{tikzcd} & \vec{x} \arrow[dl, bend right, "P_B^{-1}"'] & \\ {[\vec{x}]_B} & & {[\vec{x}]_C} \arrow[ll, "\underset{B\leftarrow C}{P}"] \arrow[ul, bend right, "P_C"'] \end{tikzcd}$$

$$P_B=\begin{bmatrix}-1 & 1\\ 4 & -3\end{bmatrix} \quad \text{and} \quad -\begin{bmatrix}-3 & -1\\ -4 & -1\end{bmatrix}=\begin{bmatrix}3 & 1\\ 4 & 1\end{bmatrix}=P_B^{-1}$$

$$P_C=\begin{bmatrix}1 & 1\\ 4 & 3\end{bmatrix}$$

$$\text{And} \quad \underset{B\leftarrow C}{P}=P_B^{-1}P_C=\begin{bmatrix}3 & 1\\ 4 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 4 & 3\end{bmatrix}=\begin{bmatrix}7 & 6\\ 8 & 7\end{bmatrix}$$

Alternate solution

$$\begin{array}{c} \quad B \qquad\quad C \\ \left[\begin{array}{cc|cc}-1 & 1 & 1 & 1\\ 4 & -3 & 4 & 3\end{array}\right] \end{array} \sim \begin{array}{c} \qquad\qquad \underset{B\leftarrow C}{P} \\ \left[\begin{array}{cc|cc}1 & 0 & 7 & 6\\ 0 & 1 & 8 & 7\end{array}\right] \end{array}$$

claim

(1.6) Let λ be an eigenvalue of an invertible matrix A. Prove that λ^{-1} is an eigenvalue of A^{-1}.

proof.

Let invertible A be given w/ eigenvalue λ.

$\Rightarrow \exists \vec{v}$ s.t. $A\vec{v} = \lambda \vec{v}$

$\Rightarrow A^{-1} A \vec{v} = \lambda A^{-1} \vec{v}$

$\Rightarrow \vec{v} = \lambda A^{-1} \vec{v}$

$\Rightarrow \frac{1}{\lambda} \vec{v} = A^{-1} \vec{v}$ ($\lambda \neq 0$ since A is invertible)

$\therefore \lambda^{-1}$ is an eigenvalue of A^{-1}.

$$A$$

(1.7) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 0 & 0 \\ -8 & 0 & -4 \end{bmatrix}$.

$\lambda = 0, -4, 7$

$\text{rref}(A - 0I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ .. so the eigenvector is

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\text{rref}(A + 4I) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ w/ eigenvec $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\text{rref}(A - 7I) = \begin{bmatrix} 1 & 0 & 11/8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ w/ eigenvec $\begin{bmatrix} -11/8 \\ 0 \\ 1 \end{bmatrix}$

(1.8) Find the eigenspace of $A = \begin{bmatrix} 6 & 0 & 1 & 0 \\ 3 & 4 & 1 & 0 \\ 4 & -3 & 3 & 0 \\ 2 & -4 & -6 & 5 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 5$.

$$\text{rref}(A - 5I) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

the eigenspace associated w/ $\lambda = 5$ is

$$\text{span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$