Name: Key
Assessment 5
Math 220: Linear Algebra

Instructions: Please carefully complete these questions by hand. Be sure to show all work (this includes notating steps in row reduction in matrices that include a variable).

Should you choose to work these on scratch paper, **please do not put more than one question on a page.** Additional sheets of paper are acceptable. Write your name on every page. **You can submit more pages, but Gradescope will not accept less pages than the original assignment.**

Upload your solutions to Gradescope by 8 am on Monday (3/1). During your presentation time, you will be asked to explain your thought process and reasoning on a randomly assigned question. Late submissions (or resubmissions) are available thru 5 pm with a 5% penalty. Resubmission is helpful if you think you can gain 5% in the process.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1TE17S-z6oWrdejMbX5txwWX-JowaTFspdJ-7pCtpodY/edit?usp=sharing

Reminders: It is okay to collaborate with peers and use online resources. However, the final work should be your own and you should be prepared to present on each question.

(1.1) Suppose $\mathbb{R}^4 = \text{Span}\{\vec{v}_1,...\vec{v}_4\}$. Explain why $\{\vec{v}_1,...\vec{v}_4\}$ is a basis for $\mathbb{R}^4$.

A minimum of 4 vectors are required to span $\mathbb{R}^4$.

We have 4 vectors that span $\mathbb{R}^4$

This is a minimal spanning set.

$\therefore$ It is a basis.

(1.2) Find a basis for the space spanned by the vectors $\begin{bmatrix} 4 \\ 9 \\ -5 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 4 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ -7 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ -15 \\ 27 \\ 9 \\ 12 \end{bmatrix}, \begin{bmatrix} -30 \\ -41 \\ -7 \\ 26 \\ -22 \end{bmatrix}$. Justify your work.

Let $A = \begin{bmatrix} 4 & 5 & -4 & 3 & -30 \\ 9 & 4 & -3 & -15 & -41 \\ -5 & 4 & -7 & 27 & -7 \\ -6 & -3 & 1 & 9 & 26 \\ 0 & 4 & -7 & 12 & -22 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -3 & -3 \\ 0 & 1 & 0 & -3 & -2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

pivots

Basis $= \left\{ \begin{bmatrix} 4 \\ 9 \\ -5 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 4 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ -7 \\ 1 \\ -7 \end{bmatrix} \right\}$

claim: (1.3) Suppose that $\{\vec{v}_1,\ldots\vec{v}_p\}$ is a subset of V and T is a one-to-one linear transformation, so that an equation $T(\vec{u})=T(\vec{v})$ always implies that $\vec{u}=\vec{v}$. Prove that if the set of images $\{T(\vec{v}_1),\ldots T(\vec{v}_p)\}$ is linearly dependent, then $\{\vec{v}_1,\ldots\vec{v}_p\}$ is linearly dependent.

proof.

Let T and $\{\vec{v}_1,\ldots,\vec{v}_p\}$ be given as above.

suppose $\{T(\vec{v}_1),\ldots,T(\vec{v}_p)\}$ is L.D.

$\Rightarrow \exists$ a non-trivial solution $c_1T(\vec{v}_1)+\ldots+c_pT(\vec{v}_p)=\vec{0}$

$\Rightarrow T(c_1\vec{v}_1)+\ldots+T(c_p\vec{v}_p)=\vec{0}$ where not all $c_1,\ldots,c_p$ are zero.

$\Rightarrow T(c_1\vec{v}_1+\ldots+c_p\vec{v}_p)=T(\vec{0})$

$\Rightarrow c_1\vec{v}_1+\ldots+c_p\vec{v}_p=\vec{0}$ w/ not all $c_1,\ldots,c_p$ zero

$\therefore \{\vec{v}_1,\ldots,\vec{v}_p\}$ are L.D.

(1.4) State the Invertible Matrix Theorem as give through section 4.5 (this requires listing assumptions and then seventeen equivalent statements).

Let A be an $n \times n$ matrix, The following are equivalent.

a) A is invertible

b) $A \sim I$

c) A has n pivots

d) $A\vec{x} = \vec{0}$ has only the trivial soln

e) The cols of A are LI.

f) The LT $\vec{x} \mapsto A\vec{x}$ is 1-1

g) The LT $\vec{x} \mapsto A\vec{x}$ is onto

h) the cols of A span $\mathbb{R}^n$

i) The LT $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

j) $\exists$ $C_{n \times n}$ s.t. $CA = I$

k) $\exists$ $D_{n \times n}$ s.t. $AD = I$

l) A^T is invertible

m) The cols of A are a basis for $\mathbb{R}^n$

n) $\text{col}(A) = \mathbb{R}^n$

o) $\text{rank}(A) = n$

p) $\text{nullity}(A) = 0$

q) $\text{null}(A) = \vec{0}$

r) $\det(A) \neq 0$.

(1.5) Let $\vec{v}_1 = \begin{bmatrix} 7 \\ -6 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -5 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 18 \\ 1 \\ 1 \end{bmatrix}$, and $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Verify that the vectors are linearly dependent and use this information to find a basis for H.

Let $A = \begin{bmatrix} 7 & 1 & 18 \\ -6 & 9 & 1 \\ 4 & -5 & 1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 7/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\text{only 2 pivots}}$

$$\frac{7}{3}\vec{v}_1 + \frac{5}{3}\vec{v}_2 = \vec{v}_3$$

Basis: $\{\vec{v}_1, \vec{v}_2\}$

(1.7) Determine whether the vectors $p_1(t) = 3 + 7t$, $p_2(t) = 4 + 2t - 3t^3$, $p_3(t) = 4t - 2t^2$, and $p_4(t) = 2 + 28t - 8t^2 + 3t^3$ form a basis for P_3. Justify your conclusions.

$\mathbb{P}_3 \Longleftrightarrow \mathbb{R}^4$. using the standard bases for $\mathbb{P}_3$ and $\mathbb{R}^4$.

$$\{p_1, p_2, p_3, p_4\} \Longleftrightarrow \left\{ \begin{bmatrix} 3 \\ 7 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 28 \\ -8 \\ 3 \end{bmatrix} \right\}$$

$$\text{Let } A = \begin{bmatrix} 3 & 4 & 0 & 2 \\ 7 & 2 & 4 & 28 \\ 0 & 0 & -2 & -8 \\ 0 & -3 & 0 & 3 \end{bmatrix}$$

$$\sim I$$

since the 4 cols of A are LI, they form a basis for $\mathbb{R}^4$ and thus $\{p_1, p_2, p_3, p_4\}$ are a basis for $\mathbb{P}_3$.

(1.8) State and prove the spanning set theorem.

Claim: Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a set in V, and let $H = \text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$

(a.) If one of the vectors in S, say $\vec{v}_k$, is a lin. comb. of the remaining vectors in S, then the set formed from S by removing $\vec{v}_k$ still spans H.

(b) If $H \neq \{\vec{0}\}$, some subset of S is a basis for H.

Proof.

prelim work: Let S, H, and $\vec{v}_k$ be as above and reindex $\vec{v}_1, \ldots, \vec{v}_p$ so $\vec{v}_k \mapsto \vec{v}_p$ and the vectors are LI until they become L.D.

a.) Let $\vec{x} \in H$ be given. $\exists\, c_1, \ldots, c_p$ s.t.

$$\vec{x} = c_1 \vec{v}_1 + \ldots + c_{p-1} \vec{v}_{p-1} + c_p \vec{v}_p$$

and $v_p = d_1 \vec{v}_1 + \ldots + d_{p-1} \vec{v}_{p-1}$ for some $d_1, \ldots, d_{p-1}$

$$\Rightarrow \vec{x} = c_1 \vec{v} + \ldots + c_{p-1} \vec{v}_{p-1} + c_p (d_1 \vec{v}_1 + \ldots + d_{p-1} \vec{v}_{p-1})$$

$$= (c_1 + c_p d_1) \vec{v}_1 + \ldots + (c_{p-1} + c_p d_{p-1}) \vec{v}_{p-1}$$

$\therefore$ The set $\vec{v}_1, \ldots, \vec{v}_{p-1}$ still spans H.

b.) By construction, the reordered $\underbrace{\vec{v}_1, \ldots, \vec{v}_\ell}_{LI}, \underbrace{\vec{v}_{\ell+1}, \ldots, \vec{v}_p}_{LD}$ are LI until they aren't.

Thus $\vec{v}_1, \ldots, \vec{v}_\ell$ are LI and spans H

$\therefore$ $\vec{v}_1, \ldots, \vec{v}_\ell$ are a basis.

(1.9) If $B=\left\{\begin{bmatrix}8\\-7\end{bmatrix},\begin{bmatrix}-2\\2\end{bmatrix}\right\}$, use two different methods to find the B-coordinates of $\vec{x}=\begin{bmatrix}4\\-2\end{bmatrix}$.

(1) $c_1\begin{bmatrix}8\\-7\end{bmatrix}+c_2\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}$

$\Rightarrow \left[\begin{array}{cc|c}8&-2&4\\-7&2&-2\end{array}\right]$

$\sim \left[\begin{array}{cc|c}1&0&2\\0&1&6\end{array}\right]$

$\Rightarrow [\vec{x}]_B=\begin{bmatrix}2\\6\end{bmatrix}$

(2) $P_T=\begin{bmatrix}8&-2\\-7&2\end{bmatrix}$

And $P_T^{-1}\vec{x}=\begin{bmatrix}2\\6\end{bmatrix}=[\vec{x}]_B$.