Name: key
Assessment 4
Math 220: Linear Algebra

Instructions: Please carefully complete these questions by hand. Be sure to show all work (this includes notating steps in row reduction in matrices that include a variable).

Should you choose to work these on scratch paper, **please do not put more than one question on a page.** Additional sheets of paper are acceptable. Write your name on every page. **You can submit more pages, but Gradescope will not accept less pages than the original assignment.**

Upload your solutions to Gradescope by 8 am on Monday (2/22). During your presentation time, you will be asked to explain your thought process and reasoning on a randomly assigned question. Late submissions (or resubmissions) are available thru 5 pm with a 5% penalty. Resubmission is helpful if you think you can gain 5% in the process.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1TE17S-z6oWrdejMbX5txwWX-JowaTFspdJ-7pCtpodY/edit?usp=sharing

Reminders: It is okay to collaborate with peers and use online resources. However, the final work should be your own and you should be prepared to present on each question.

(1.1) Let $A = \begin{bmatrix} -2 & -4 & -16 \\ 3 & 5 & 19 \\ 1 & 3 & 14 \end{bmatrix}$. Find the third column of A^{-1} without computing the other two columns. Explain how this works. (Give exact answers)

$$\left[\begin{array}{ccc|c} -2 & -4 & -16 & 0 \\ 3 & 5 & 19 & 0 \\ 1 & 3 & 14 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

so the 3^{rd} col. of A^{-1} is $\begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$

This works because solving $A\vec{x} = \vec{e}_3$ gives the 3^{rd} col of A^{-1}!

(1.2) Find the inverse of $A = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 11 & -6 \\ -4 & -1 & 5 \end{bmatrix}$, if it exists. Use the algorithm for finding A^{-1} that involves row reducing $[A \mid I]$. If A is invertible, explain why this means the columns of A are linearly independent.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ -5 & 11 & -6 & 0 & 1 & 0 \\ -4 & -1 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1/49 & -11/49 \\ 0 & 1 & -1 & 0 & 4/49 & -5/49 \\ 0 & 0 & 0 & 1 & 9/49 & 1/49 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{matrix}}_{\neq I}$$

so A is <u>not</u> invertible and its columns are <u>not</u> linearly independent.

(1.3) Let A and B be $n \times n$ matrices. Prove that if AB is invertible, so is B.

Use the proof structure taught in class, in the videos, and in the class notes.

claim: If A and B are $n \times n$ matrices and AB is invertible, then B is invertible.

proof.

Let A and B are as above.

AB is invertible

$\Rightarrow \det(AB) \neq 0$

$\Rightarrow \det(A)\det(B) \neq 0$

$\Rightarrow \det(B) \neq 0$

$\Rightarrow B$ is invertible.

Q.E.D

(1.4) Consider the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ where $T(x_1,x_2)=(2x_1-7x_2,-2x_1+6x_2)$. Show that T is invertible and find a formula for T^{-1}. Your result should be in the same notation as the T provided.

$T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} 2 & -7 \\ -2 & 6 \end{bmatrix}$

$\Rightarrow A^{-1} = -\frac{1}{2}\begin{bmatrix} 6 & 7 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -7/2 \\ -1 & -1 \end{bmatrix}$

$T^{-1}(x_1, x_2) = (-3x_1 \qquad , -x_1 - x_2)$

(1.5) By hand, compute $\begin{vmatrix} -1 & 3 & 6 & 0 \\ 4 & 5 & 3 & 0 \\ 4 & 4 & 6 & 8 \\ 4 & 2 & 4 & 4 \end{vmatrix}$. (Checking with a calculator is recommended).

Is the associated matrix invertible? why/why not.

$$\begin{vmatrix} -1 & 3 & 6 & 0 \\ 4 & 5 & 3 & 0 \\ 4 & 4 & 6 & 8 \\ 4 & 2 & 4 & 4 \end{vmatrix} = -8\begin{vmatrix} -1 & 3 & 6 \\ 4 & 5 & 3 \\ 4 & 2 & 4 \end{vmatrix} + 4\begin{vmatrix} -1 & 3 & 6 \\ 4 & 5 & 3 \\ 4 & 4 & 6 \end{vmatrix}$$

$$= -8\left[-1\begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 4 & 4 \end{vmatrix} + 6\begin{vmatrix} 4 & 5 \\ 4 & 2 \end{vmatrix}\right]$$

$$+ 4\left[-1\begin{vmatrix} 5 & 3 \\ 4 & 6 \end{vmatrix} - 3\begin{vmatrix} 4 & 3 \\ 4 & 6 \end{vmatrix} + 6\begin{vmatrix} 4 & 5 \\ 4 & 4 \end{vmatrix}\right]$$

$$= -8\left[\underbrace{-1(14) - 3(4) + 6(-12)}_{-98}\right] + 4\left[\underbrace{-1(18) - 3(12) + 6(-4)}_{-78}\right]$$

$$= 472$$

Since det of the associated matrix is non-zero, the matrix is invertible.

(1.6) <u>By hand</u>, compute $\det\left(B^3\right)$ where $B = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$. (<u>Checking with a calculator is recommended</u>).

$$\det(B^3) = [\underbrace{\det(B)}]^3$$

$$2\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}$$

$$= [0 - 2(6)]^3$$

$$= -1728$$

(1.7) Let H and K be subspaces of a vector space V. The intersection if H and K, written as $H \cap K$, is the set of $\vec{v} \in V$ that belong to both H and K. Prove that $H \cap K$ is a subspace of V.

Use the proof structure taught in class, in the videos, and in the class notes.

Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

claim: If H, K are subspaces of vector space V, then $H \cap K$ is a subspace.

proof

Let H, k, and V be as given above.

(1) $\vec{0} \in H$ and $\vec{0} \in k$ so $\vec{0} \in H \cap k$

(2) suppose $\vec{u}, \vec{w} \in H \cap k$

$\Rightarrow \vec{u} + \vec{w} \in H$ and $\vec{u} + \vec{w} \in k$

$\Rightarrow u + w \in H \cap k$

(3) suppose $\vec{u} \in H \cap k$ and $r \in \mathbb{R}$

$\Rightarrow r\vec{u} \in H$ and $r\vec{u} \in k$

$\Rightarrow r\vec{u} \in H \cap k$

$\therefore H \cap k$ is a subspace.

$H \cup k$ is not generally a subspace. For example, let $H = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $k = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin H \cup k$ but is clearly the sum of vectors in $H \cup k$, (1.8) The set of $M_{2\times2}$ of all 2×2 matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars. Prove that the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is (or isn't) a subspace of $M_{2\times2}$.

Use the proof structure taught in class, in the videos, and in the class notes.

Claim: the set of all $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace.

proof

1. If $a = b = d = 0$, then $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = 0_{2\times2}$

so $0_{2\times2}$ is in the space

2. $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ 0 & h \end{bmatrix}$ are in the space, Now $A + B = \begin{bmatrix} a+e & b+f \\ 0 & d+h \end{bmatrix}$. This has the same form and so is also in the space.

3. $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $c \in \mathbb{R}$ are given.

$rA = \begin{bmatrix} ra & rb \\ 0 & rd \end{bmatrix}$ is of the same form as A. so in the space

$\therefore$ It is a subspace.

$$\vec{v}$$

(1.9) Determine whether the vector $\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$ is in the column space of $A = \begin{bmatrix} -11 & 7 & 1 & 0 \\ -5 & 2 & 4 & 5 \\ 10 & -8 & 4 & 7 \\ 3 & -2 & 0 & 0 \end{bmatrix}$, the null space of A, both, or neither. Explain/justify your reasoning.

col. space

$$\left[A \,\middle|\, \begin{matrix} 2 \\ 3 \\ 1 \\ 0 \end{matrix} \right] \sim \left[\begin{matrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \,\middle|\, \begin{matrix} -4 \\ -6 \\ -1 \\ 0 \end{matrix} \right]$$

the system is consistent ($A\vec{x} = \vec{v}$) so $\vec{v} \in$ col A.

null space

$$A\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{so} \quad \vec{v} \in \text{Null}(A)$$

(1.10) Let $T: V \to W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W.

Use the proof structure taught in class, in the videos, and in the class notes.

claim: $T: V \Rightarrow W$ is a linear transformation so range(T) is a subspace of W.

proof.

① $T(\vec{0}) = \vec{0} \in$ range (T)

② $\vec{u}, \vec{v} \in$ range of $T \Rightarrow \exists\, \vec{a}, \vec{b}$ s.t. $T(\vec{a}) = \vec{u}$ and $T(\vec{b}) = \vec{v}$

$\Rightarrow T(\vec{a}+\vec{b}) = T(\vec{a}) + T(\vec{b}) = \vec{u} + \vec{v} \in$ range (T)

③ $\vec{u} \in$ range (T) and $r \in \mathbb{R}$

$\Rightarrow \exists\, \vec{a}$ s.t. $T(\vec{a}) = \vec{u}$

$\Rightarrow T(r\vec{a}) = r\,T(\vec{a}) = r\,\vec{u} \in$ range (T)

$\therefore$ T is a subspace.