Name: Key
Assessment 3
Math 220: Linear Algebra

Instructions: Please carefully complete these questions by hand. Be sure to show all work (this includes notating steps in row reduction in matrices that include a variable).

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. Write your name on every page. You can submit more pages, but Gradescope will not accept less pages than the original assignment.

Upload your solutions to Gradescope by 8 am on Monday (2/8). During your presentation time, you will be asked to explain your thought process and reasoning on a randomly assigned question. Late submissions (or resubmissions) are available thru 5 pm with a 5% penalty. Resubmission is helpful if you think you can gain 5% in the process.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1TE17S-z6oWrdejMbX5txwWX-JowaTFspdJ-7pCtpodY/edit?usp=sharing

Reminders: It is okay to collaborate with peers and use online resources. However, the final work should be your own and you should be prepared to present on each question.

(1.1) Let $\vec{b}=\begin{bmatrix}5\\19\\41\\2\end{bmatrix}$ and $A=\begin{bmatrix}4&-2&5&-9\\-5&7&-8&0\\-8&10&5&3\\6&-4&8&-11\end{bmatrix}$. Is $\vec{b}$ in the range of the transformation $\vec{x}\to A\vec{x}$. If so, find all $\vec{x}$ whose image under the transformation is $\vec{b}$. Justify your answer.

solve $A\vec{x}=\vec{b}$

$$\left[\begin{array}{cccc|c}4&-2&5&-9&5\\-5&7&-8&0&19\\-8&10&5&3&41\\6&-4&8&-11&2\end{array}\right]\sim\left[\begin{array}{cccc|c}1&0&0&-3.5&3\\0&1&0&-2.5&6\\0&0&1&0&1\\0&0&0&0&0\end{array}\right]$$

Yes, $\vec{x}=\begin{bmatrix}3\\6\\1\\0\end{bmatrix}+x_4\begin{bmatrix}3.5\\2.5\\0\\1\end{bmatrix}$

(1.2) Let $T:\mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Prove that the set $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent.

Use the proof structure taught in class, in the videos, and in the class notes.

claim: If $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Then $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is linearly dependent.

proof.

Let T and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be as above.

$\Rightarrow \exists\, a,b,c \in \mathbb{R}$ (not all 0) s.t. $a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3 = \vec{0}$

$\Rightarrow \underbrace{T(a\vec{v}_1 + b\vec{v}_2 + c\vec{v}_3)} = \underbrace{T(\vec{0})}$

$\Rightarrow T(av_1) + T(bv_2) + T(cv_3) = \vec{0}$

$\Rightarrow aT(\vec{v}_1) + bT(\vec{v}_2) + c\,T(\vec{v}_3) = \vec{0}$

$\Rightarrow \{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ are L.D.

Q.E.D.

(1.3) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{e}_1 \to \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\vec{e}_2 \to \begin{bmatrix} -2 \\ 8 \end{bmatrix}$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and find the matrix of the linear transformation.

$$T(\vec{x}) = A\vec{x} \quad \text{where} \quad A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -2 \\ 5 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ -3 \end{bmatrix} \longmapsto \begin{bmatrix} 4 & -2 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 26 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto \begin{bmatrix} 4 & -2 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\begin{bmatrix} 4 \\ 5 \end{bmatrix} + x_2\begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

(1.4) Let T be the linear transformation matrix with standard matrix $A = \begin{bmatrix} 4 & -7 & 3 & -1 & 5 \\ 6 & -8 & 5 & 0 & -8 \\ -7 & 10 & -8 & 5 & 14 \\ 3 & -5 & 4 & -4 & -6 \\ -5 & 6 & -6 & 3 & 3 \end{bmatrix}$.

Is the transformation <u>one-to-one</u>? Is it <u>onto</u>? Justify your answers.

$$A \sim \begin{bmatrix} \boxed{1} & 0 & 0 & 3 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Four pivots.

Using Theorem 8 in 2,3, since $A_{5\times5}$ is not invertible we know that the linear transformation is <u>not</u> 1-1 and not <u>onto</u>.

(1.5) Describe the possible echelon forms of the standard matrix for a linear transformation T where $T:\mathbb{R}^3 \to \mathbb{R}^5$. Please make sure to clearly define any special characters that you use.

$T(\vec{x}) = A\vec{x}$ where $A_{5\times 3}$

onto:

$$\begin{bmatrix} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & \blacksquare \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{5\times 3}$$

where $\blacksquare$ is any non-zero number, $*$ is any number

1 to 1:

None, since it takes at least 5 cols in A to hit every point in $\mathbb{R}^5$. Thus the image is a proper subset of $\mathbb{R}^5$.

(1.6) Determine if the linear transformation $T(x_1, x_2 x_3) = (x_1 - 5x_2 + x_3, x_2 - 7x_3)$ is <u>one-to-one</u> and <u>onto</u>. Justify your answer.

$T(\vec{x}) = A\vec{x}$ $\quad T(\vec{e}_1) \quad T(\vec{e}_2) \quad T(\vec{e}_3)$

$$\text{w/ } A = \begin{bmatrix} 1 & -5 & 1 \\ 0 & 1 & -7 \end{bmatrix}_{2 \times 3}$$

T is <u>onto</u> because the columns span $\mathbb{R}^2$.

T is <u>not 1-1</u> because to cols are not L.I.

(1.7) If $\vec{u} = \begin{bmatrix} -9 \\ 2 \\ -6 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, compute $\vec{u}^T\vec{v}$, $\vec{v}^T\vec{u}$, $\vec{u}\vec{v}^T$, $\vec{v}\vec{u}^T$.

$$\vec{u}^T\vec{v} = \begin{bmatrix} -9 & 2 & -6 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = -9a + 2b - 6c$$

$$\vec{v}^T\vec{u} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -9 \\ 2 \\ -6 \end{bmatrix} = -9a + 2b - 6c$$

$$\vec{u}\vec{v}^T = \begin{bmatrix} -9 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -9a & -9b & -9c \\ 2a & 2b & 2c \\ -6a & -6b & -6c \end{bmatrix}$$

$$\vec{v}\vec{u}^T = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} -9 & 2 & -6 \end{bmatrix} = \begin{bmatrix} -9a & 2a & -6a \\ -9b & 2b & -6b \\ -9c & 2c & -6c \end{bmatrix}$$

(1.8) If $A = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 \\ -2 & k \end{bmatrix}$, what value(s) of k, if any, will make $AB = BA$.

$$AB = \begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 6 \\ -2 & k \end{bmatrix} = \begin{bmatrix} 2 & 24+3k \\ -6 & -6+2k \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 6 \\ -2 & k \end{bmatrix}\begin{bmatrix} 4 & 3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 18 \\ -8-k & -6+2k \end{bmatrix}$$

$\Rightarrow 24+3k = 18 \Rightarrow 3k = -6 \Rightarrow k = -2$

and $-6 = -8-k \Rightarrow 2 = -k \Rightarrow k = -2$

so $AB = BA$ when $k = -2$.

has only the trivial solution

(1.9) Prove that if $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(\vec{x})=\vec{0}$ if and only if T is one-to-one.

Use the proof structure taught in class, in the videos, and in the class notes.

Claim: If $T:\mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T(\vec{x})=\vec{0}$ has only the trivial solution iff T is 1-1.

($\Rightarrow$) Assume $T(\vec{x})=\vec{0}$ has only the trivial solution. Let's suppose T is not 1-1,

$\Rightarrow \exists\ \vec{u}\neq\vec{v}$ and $\vec{b}$ s.t. $T(\vec{u})=\vec{b}$ and $T(\vec{v})=\vec{b}$

$\Rightarrow T(\vec{u}-\vec{v}) = T(\vec{u})-T(\vec{v}) = \vec{b}-\vec{b}=\vec{0}$

$\Rightarrow \vec{u}-\vec{v}=\vec{0}$

$\Rightarrow \vec{u}=\vec{v} \Rightarrow \Leftarrow$

$\Rightarrow T$ is 1-1.

($\Leftarrow$) Assume T is 1-1.

we know $T(\vec{0})=\vec{0}$ has a solution and it must be unique since T is 1-1.

Q.E.D

(1.10) Explain how you would show a transformation is a linear transformation. Provide a mathematical example of a <u>non</u>-linear transformation (show that it does not meet the necessary conditions).

To show T is linear, I would take two arbitrary elements in the domain and a scalar and show

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$

(b) $T(a\vec{u}) = a\,T(\vec{u})$.

an example of a non L.T.

$T: x \longmapsto x^2$.

(a) $T: (1+1) \longmapsto 4$

but $T(1) + T(1) = 2$.