Name: Key

Assessment 1

Math 220: Linear Algebra

Instructions: Please carefully complete these questions by hand. Be sure to show all work (this includes notating steps in row reduction for sections 1.1, 1.2, and in matrices that include a variable).

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. Write your name on every page. You can submit more pages, but Gradescope will not accept less pages than the original assignment.

Upload your solutions to Gradescope by 8 am on Monday (1/25). During your presentation time, you will be asked to explain your thought process and reasoning on a randomly assigned question. Late submissions (or resubmissions) are available thru 5 pm with a 5% penalty. Resubmission is helpful if you think you can gain 5% in the process.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1TE17S-z6oWrdejMbX5txwWX-JowaTFspdJ-7pCtpodY/edit?usp=sharing

Reminders: It is okay to collaborate with peers and use online resources. However, the final work should be your own and you should be prepared to present on each question.

(1.1) Determine the value(s) of h such that $\left[\begin{array}{cc|c} 1 & h & 4 \\ 5 & 15 & 16 \end{array}\right]$ is the augmented matrix of an inconsistent system. What is the solution to this system for this value(s) of h?

$$\left[\begin{array}{cc|c} 1 & h & 4 \\ 5 & 15 & 16 \end{array}\right] \quad R_2 - 5R_1 \rightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & h & 4 \\ 0 & 15-5h & -4 \end{array}\right]$$

The system is inconsistent when $15 - 5h = 0$ or $h = 3$.

so the system will have no solution @ $h = 3$.

(1.2) Using the methods developed in this course, solve the system

$$\begin{aligned} x_1 \quad\quad\; - 3x_3 &= 6 \\ 2x_1 + 4x_2 + 7x_3 &= 19. \\ 2x_2 + 4x_3 &= 6 \end{aligned}$$

Be sure to notate all steps using the notation from class and/or the text.

Solve $\left[\begin{array}{ccc|c} 1 & 0 & -3 & 6 \\ 2 & 4 & 7 & 19 \\ 0 & 2 & 4 & 6 \end{array}\right]$ $\begin{array}{l} R_2 - 2R_1 \to R_2 \\ \frac{1}{2}R_3 \to R_3 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 6 \\ 0 & 4 & 13 & 7 \\ 0 & 1 & 2 & 3 \end{array}\right]$ $R_2 \leftrightarrow R_3$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 13 & 7 \end{array}\right]$ $R_3 - 4R_2 \to R_3$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & -5 \end{array}\right]$ $\frac{1}{5}R_3 \to R_3$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$ $\begin{array}{l} R_1 + 3R_3 \to R_1 \\ R_2 - 2R_3 \to R_2 \end{array}$

$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right]$

$x_1 = 3$; $x_2 = 5$; $x_3 = -1$

(1.3) Consider the system $\begin{array}{rcrcl} x_1 & + & hx_2 & = & 5 \\ 5x_1 & + & 10x_2 & = & k \end{array}$. Choose h and k such that the system has:

$$\left[\begin{array}{cc|c} 1 & h & 5 \\ 5 & 10 & k \end{array}\right] \quad R_2 - 5R_1 \rightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & h & 5 \\ 0 & 10-5h & k-25 \end{array}\right] \quad \frac{1}{10-5h} R_2 \rightarrow R_2$$

$$\sim \left[\begin{array}{cc|c} 1 & h & 5 \\ 0 & 1 & \frac{k-25}{10-5h} \end{array}\right] \quad R_1 - hR_2 \rightarrow R_1$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & 0 & 5 - h\frac{k-25}{10-5h} \\ 0 & 1 & \frac{k-25}{10-5h} \end{array}\right]$$

(a.) No solution

$h = 2$ AND $k \neq 25$

(b.) A unique solution. Find the solution.

$h \neq 2$

$$x_1 = 5 - h\frac{k-25}{10-5h}$$

$$x_2 = \frac{k-25}{10-5h}$$

(c.) Many solutions. Find these solutions. (write in vector form)

$h = 2$ AND $k = 25$

$$\left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 0 & 0 \end{array}\right] \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$x_1 + 2x_2 = 5$

x_2 free.

(1.4) Find the general solution $\bar{x}$ of the system with augmented matrix $\left[\begin{array}{ccccc|c} 1 & -9 & 0 & -1 & 0 & -6 \\ 0 & 1 & 0 & 0 & -8 & 1 \\ 0 & 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

Express your solution in the vector form we used in examples 1, 2, 4, and 5 in section 1.5.

$$\left[\begin{array}{ccccc|c} 1 & -9 & 0 & -1 & 0 & -6 \\ 0 & 1 & 0 & 0 & -8 & 1 \\ 0 & 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad R_1 + 9R_2 + R_3 \to R_1$$

$$\sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -65 & 8 \\ 0 & 1 & 0 & 0 & -8 & 1 \\ 0 & 0 & 0 & 1 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8 + 65x_5 \\ 1 + 8x_5 \\ x_3 \\ 5 - 7x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 65 \\ 8 \\ 0 \\ -7 \\ 1 \end{bmatrix}$$

$\vec{b}$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \vec{A}$

(1.5) Determine if $\begin{bmatrix} 10 \\ -5 \\ 6 \end{bmatrix}$ is a linear combination of the columns of $\begin{bmatrix} 1 & -4 & -4 \\ 0 & 5 & 7 \\ 4 & -16 & 15 \end{bmatrix}$. If you use a calculator, please make sure to clearly show inputs/outputs from the calculator. Explain.

solve $\left[\begin{array}{ccc|c} 1 & -4 & -4 & 10 \\ 0 & 5 & 7 & -5 \\ 4 & -16 & 15 & 6 \end{array}\right]$

RREF ON TI84 $\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1202/155 \\ 0 & 1 & 0 & 83/155 \\ 0 & 0 & 1 & -34/31 \end{array}\right]$

Yes, we found $\frac{1202}{155}\vec{a}_1 + \frac{83}{155}\vec{a}_2 - \frac{34}{31}\vec{a}_3 = \vec{b}$.

(1.6) Consider the system of equations $\begin{array}{rcrcrcl} & & x_2 & + & 4x_3 & = & 0 \\ 4x_1 & + & 7x_2 & - & x_3 & = & 0 \\ -x_1 & + & 3x_2 & - & 8x_3 & = & 0 \end{array}$. Express this system in the following three forms:

a.) As a matrix equation.

$$\underset{A}{\begin{bmatrix} 0 & 1 & 4 \\ 4 & 7 & -1 \\ -1 & 3 & -8 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

b.) As a vector equation.

$$\vec{x}_1 \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -1 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

c.) As an augmented matrix.

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 4 & 7 & -1 & 0 \\ -1 & 3 & -8 & 0 \end{array}\right]$$

$12i + 6i^2$

$4i +$

$-36 + -45i + -18i + -15i^2$

(1.7) For what values of h, is $\begin{bmatrix} 4 \\ 6 \\ h \end{bmatrix}$ in the span of $\begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ -18 \\ 2 \end{bmatrix}$?

$$\left[\begin{array}{cc|c} 1 & -5 & 4 \\ 5 & -18 & 6 \\ -1 & 2 & h \end{array}\right] \begin{array}{l} R_2 - 5R_1 \to R_2 \\ R_3 + R_1 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -5 & 4 \\ 0 & 7 & -14 \\ 0 & -3 & 4+h \end{array}\right] \begin{array}{l} \frac{1}{7}R_2 \to R_2 \\ 7R_3 + 3R_2 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & -5 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & -14+7h \end{array}\right]$$

It is in the span so long as the system is consistent. That is, so long as $h = 2$.

$$\begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array}$$

(1.8) Do the vectors $\left\{ \begin{bmatrix} 0 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ -6 \\ 15 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -10 \end{bmatrix} \right\}$ span $\mathbb{R}^3$? Why or why not? If you use a calculator, please make sure to clearly show inputs/outputs from the calculator.

solve $A\vec{x} = \vec{b}$ for an arbitrary $\vec{b}$ w/ $A = [\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3]$

$$\left[\begin{array}{ccc|c} 0 & 0 & 3 & b_1 \\ 0 & -6 & -2 & b_2 \\ -5 & 15 & -10 & b_3 \end{array}\right] \quad R_1 \leftrightarrow R_3$$

$$\sim \left[\begin{array}{ccc|c} \boxed{-5} & 15 & -10 & b_3 \\ 0 & \boxed{-6} & -2 & b_2 \\ 0 & 0 & \boxed{3} & b_1 \end{array}\right]$$

There are pivots in each row

Hence the vectors span $\mathbb{R}^3$.

(1.9) Prove that for all $\vec{u} \in \mathbb{R}^n$ and scalars c and d, $(c+d)\vec{u} = c\vec{u} + d\vec{u}$. Use the notation and structure taught in class (also available on my website, posted class notes, and in the videos).

claim: $(c+d)\vec{u} = c\vec{u} + d\vec{u}$

proof.

Let $\vec{u} \in \mathbb{R}^n$ and scalars c, d be given.

$$(c+d)\vec{u} = (c+d)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{where } \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$= \begin{bmatrix} (c+d)u_1 \\ \vdots \\ (c+d)u_n \end{bmatrix}$$

$$= \begin{bmatrix} cu_1 + du_1 \\ \vdots \\ cu_n + du_n \end{bmatrix}$$

$$= \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} + \begin{bmatrix} du_1 \\ \vdots \\ du_n \end{bmatrix}$$

$$= c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + d\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

$$= c\vec{u} + d\vec{u}$$

$\therefore (c+d)\vec{u} = c\vec{u} + d\vec{u}$.

(1.10) Prove that if A is an $m \times n$ matrix and $\vec{u}, \vec{v} \in \mathbb{R}^n$, then $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$. Use the notation and structure taught in class (also available on my website, posted class notes, and in the videos).

Claim: If $A_{m \times n}$ and $\vec{u}, \vec{v} \in \mathbb{R}^n$, then $A(u+v) = A\vec{u} + A\vec{v}$.

proof.

Let $A_{m \times n}$ w/ columns $\vec{a}_1, \ldots, \vec{a}_n$ and $\vec{u}, \vec{v} \in \mathbb{R}^n$ be given.

$$A(\vec{u}+\vec{v}) = A\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}\right) \text{ where } \vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$$

$$= (u_1 + v_1)\vec{a}_1 + \ldots + (u_n + v_n)\vec{a}_n$$

$$= u_1\vec{a}_1 + v_1\vec{a}_1 + \ldots + u_n\vec{a}_n + v_n\vec{a}_n$$

$$= u_1\vec{a}_1 + \ldots + u_n\vec{a}_n + v_1\vec{a}_1 + \ldots + v_n\vec{a}_n$$

$$= \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$= A\vec{u} + A\vec{v}$$

$\therefore A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$.