Name: ________________________
Final Exam Assessment
Math 220: Linear Algebra

Instructions: Please carefully complete these questions by hand. Be sure to show all work (this includes notating steps in row reduction in matrices that include a variable).

Should you choose to work these on scratch paper, please do not put more than one question on a page. Additional sheets of paper are acceptable. Write your name on every page. You can submit more pages, but Gradescope will not accept less pages than the original assignment.

Upload your solutions to Gradescope by 8 am on Tuesday (3/23). During your presentation time, you will be asked to explain your thought process and reasoning on a randomly assigned question. Late submissions (or resubmissions) are available thru Thursday (3/25) at 5 pm with a 5% penalty. Resubmission is helpful if you think you can gain 5% in the process.

Please make sure to sign up for your presentation slot. If you are unavailable for any of the times available, please send me a note in Slack and we will find a time that works for you.

https://docs.google.com/spreadsheets/d/1TE17S-z6oWrdejMbX5txwWX-JowaTFspdJ-7pCtpodY/edit?usp=sharing

Reminders: It is okay to collaborate with peers and use online resources. However, the final work should be your own and you should be prepared to present on each question.

(1.1) A is a 3×3 with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why or why not? Please explain.

(1.2) Diagonalize the matrix $A = \begin{bmatrix} -1 & -12 & 6 \\ 3 & 11 & -3 \\ 9 & 18 & -4 \end{bmatrix}$. Please show the setup of any equations, but you do not need to show basic algebraic steps.

(1.3) In P_2, find the change-of-coordinates matrix from basis $B = \{1-3t+t^2, 2-5t+3t^2, 2-3t+6t^2\}$ to the standard basis $C = \{1, t, t^2\}$. Then find the *B*-coordinate vector for $9t-4-6t^2$.

Please clearly describe any isomorphism between P_2 and $\mathbb{R}^3$ by showing what happens to an arbitrary element under the transformation.

(1.4) Let $T : V \to W$ be a linear transformation from a vector space V to a vector space W. Prove that the range of T is a subspace of W.

(1.5) Use combine multiple methods to compute $\begin{vmatrix} -1 & 3 & 7 & 0 \\ 3 & 2 & 4 & 0 \\ 7 & 4 & 8 & 4 \\ 5 & 2 & 5 & 2 \end{vmatrix}$. Do your work by hand (although you may check with a calculator). Note: You only to find the determinant once … just combine methods in your solutions.

(1.6) By hand, find the inverse of $A = \begin{bmatrix} 1 & -2 & 1 \\ -5 & 11 & -6 \\ 3 & -8 & 5 \end{bmatrix}$. Please check with technology.

(1.7) Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $T:\begin{bmatrix}3\\5\end{bmatrix}\to\begin{bmatrix}5\\1\end{bmatrix}$ and $T:\begin{bmatrix}4\\4\end{bmatrix}\to\begin{bmatrix}-1\\4\end{bmatrix}$. Find the image of $2\begin{bmatrix}4\\4\end{bmatrix}+3\begin{bmatrix}3\\5\end{bmatrix}$.

(1.8) A certain experiment produces the data (1, 1.6), (2, 2.7), (3, 3.1), (4, 3.6), and (5, 4.1). Describe the model that produces a least-squares fit of these points by a function of the form $y = \beta_1 x + \beta_2 x^2$.

(1.9) Using the methods of this course, find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the given data points (5, 6), (6, 4), (8, 2), and (9, 0).

(1.10) Find the orthogonal projection of $\vec{b} = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 5 \end{bmatrix}$ onto Col A where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$. Find a least-squares solution of $A\vec{x} = \vec{b}$.

(1.11) Prove that if $S = \{\vec{u}_1, ..., \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then *S* is linearly independent and hence is a basis for the subspace spanned by *S*.

(1.12) For $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 7 \\ 14 \\ 0 \end{bmatrix}$, a least-squares solution of $A\vec{x} = \vec{b}$ is $\hat{x} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$. Compute the least-squares error associated with this solution and sketch a picture that will help you explain the meaning of the error.

(1.13) Find an orthogonal basis for the column space of $\begin{bmatrix} -9 & -7 & 0 & 9 \\ 1 & 4 & 4 & 1 \\ -6 & -8 & -1 & 4 \\ 15 & 15 & 22 & 29 \\ 1 & 4 & 4 & 13 \end{bmatrix}$.

(1.14) Find an orthogonal basis for the image of the transformation whose transformation matrix is given by the matrix $\begin{bmatrix} 1 & 4 & 5 \\ -1 & -4 & 1 \\ 0 & 2 & 4 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}$.

(1.15) Show that the vectors $\begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$ are orthogonal. Using these two vectors and a third of your own making, construct an orthonormal basis for $\mathbb{R}^3$.

(1.16) Find the distance from the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix} 1 \\ -1 \\ -1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 1 \\ 0 \\ 3 \end{bmatrix}$ to the vector $\begin{bmatrix} 11 \\ -3 \\ -4 \\ -5 \end{bmatrix}$.

(1.17) Prove that the orthogonal projection of a vector onto a line in $\mathbb{R}^2$ through the origin does not depend upon the choice of non-zero vectors parallel to the line.

(1.18) Explain why a square matrix with orthonormal columns is invertible.

(1.19) Let $\vec{u}=\begin{bmatrix}-6\\5\\-1\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}4\\3\\-9\end{bmatrix}$. Using the methods of this course, compute and compare $\vec{u}\cdot\vec{v}, \|\vec{u}\|^2, \|\vec{v}\|^2$ and $\|\vec{u}+\vec{v}\|^2$.

(1.20) Derive the method for finding a least-squares solution.