

6.2: Orthogonal Sets

Math 220: Linear Algebra

A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is called an orthogonal set if each pair of distinct vectors from the set is orthogonal. That is, $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ when $i \neq j$.

Ex 1: Determine whether the set of vectors is orthogonal.

a) $\begin{matrix} \vec{u}_1 \\ \begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix} \end{matrix}, \begin{matrix} \vec{u}_2 \\ \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix} \end{matrix}, \begin{matrix} \vec{u}_3 \\ \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} \end{matrix}$

$$\vec{u}_1 \cdot \vec{u}_2 = -12 + 21 - 9 = 0 \checkmark$$
$$\vec{u}_1 \cdot \vec{u}_3 = 6 - 7 + 1 = 0 \checkmark$$
$$\vec{u}_2 \cdot \vec{u}_3 = -18 - 3 - 9 = -30 \neq 0$$

This is not an orthogonal set.

b) $\begin{matrix} \vec{v}_1 \\ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix} \end{matrix}, \begin{matrix} \vec{v}_2 \\ \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix} \end{matrix}, \begin{matrix} \vec{v}_3 \\ \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \end{matrix}$

$$\vec{v}_1 \cdot \vec{v}_2 = -3 - 6 - 3 + 12 = 0$$
$$\vec{v}_1 \cdot \vec{v}_3 = 9 - 16 + 7 + 0 = 0$$
$$\vec{v}_2 \cdot \vec{v}_3 = -3 + 24 - 21 + 0 = 0$$

This is an orthogonal set.

Theorem 4

If $S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is linearly independent and hence is a basis for the subspace spanned by S .

Proof:

$$\text{solve } \vec{0} = c_1 \vec{u}_1 + \dots + c_p \vec{u}_p$$

$$\text{consider } \vec{u}_i \cdot \vec{0} = \vec{u}_i \cdot (c_1 \vec{u}_1 + \dots + c_p \vec{u}_p) \text{ for } i=1, \dots, p$$

$$\Rightarrow 0 = c_i \underbrace{\vec{u}_i \cdot \vec{u}_i}$$

$$\Rightarrow 0 = c_i \text{ for } i=1, \dots, p$$

$\Rightarrow \vec{u}_1, \dots, \vec{u}_p$ are linearly independent and clearly span the subspace.

$\therefore S$ is a basis.

6.2: Orthogonal Sets

Definition

An **orthogonal basis** for a subspace W of \mathbb{R}^n is a basis for W that is also an orthogonal set.

Theorem 5

Let $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of \mathbb{R}^n . For each \mathbf{y} in W , the weights in the linear combination

$$\mathbf{y} = c_1 \mathbf{u}_1 + \dots + c_p \mathbf{u}_p$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \dots, p)$$

To see this,
dot both sides
with \vec{u}_j
 $\vec{y} \cdot \vec{u}_j = c_1 \vec{u}_1 \cdot \vec{u}_j + \dots + c_p \vec{u}_p \cdot \vec{u}_j$

$= 0 + \dots + c_j \vec{u}_j \cdot \vec{u}_j + 0 + \dots + 0$

Ex 2: The vector $\mathbf{v} = \begin{bmatrix} 4 \\ -8 \\ -10 \\ 17 \end{bmatrix}$ is in the subspace W with orthogonal basis from Ex 1b).

Express \mathbf{v} as a linear combination of the orthogonal basis.

$$\begin{bmatrix} \vec{u}_1 \\ 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} \vec{u}_2 \\ -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} \vec{u}_3 \\ 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$$

Find $\vec{v} = c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3$

$$c_1 = \frac{\vec{v} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} = \frac{12 + 16 - 10 + 51}{9 + 4 + 1 + 9} = \frac{69}{23} = 3$$

$$c_2 = \frac{\vec{v} \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2} = \frac{-4 - 24 + 30 + 68}{1 + 9 + 9 + 16} = 2$$

$$c_3 = \frac{\vec{v} \cdot \vec{u}_3}{\vec{u}_3 \cdot \vec{u}_3} = \frac{12 - 64 - 70 + 0}{9 + 64 + 49 + 0} = -1$$

$$\Rightarrow \vec{v} = 3\vec{u}_1 + 2\vec{u}_2 - \vec{u}_3.$$

6.2: Orthogonal Sets

An Orthogonal Projection

$$\vec{y} = \hat{y} + \vec{z}$$

\uparrow \uparrow
 in W \perp to W

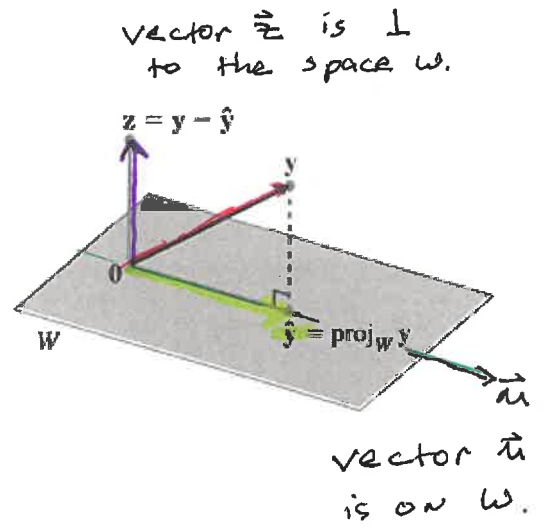
$$\Rightarrow \vec{y} = \alpha \vec{u} + \vec{z}$$

$$\Rightarrow \vec{z} = \vec{y} - \alpha \vec{u}$$

$$\Rightarrow \vec{z} \cdot \vec{u} = (\vec{y} - \alpha \vec{u}) \cdot \vec{u}$$

$$\Rightarrow 0 = \vec{y} \cdot \vec{u} - \alpha \vec{u} \cdot \vec{u}$$

$$\Rightarrow \alpha = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \quad \text{Thus} \quad \hat{y} = \frac{\vec{y} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \quad \text{AND} \quad \vec{z} = \vec{y} - \hat{y}$$



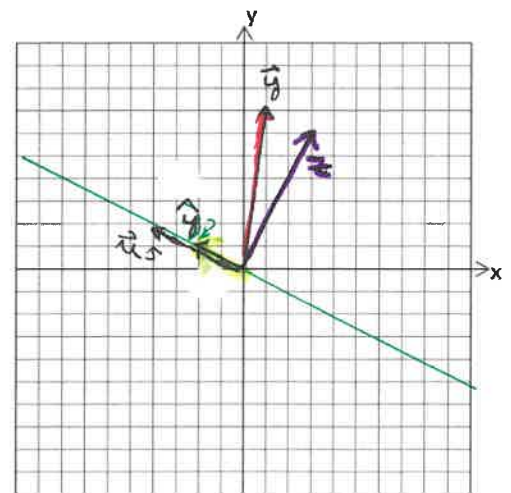
$$\hat{y} = \text{proj}_L y = \frac{y \cdot u}{u \cdot u} u$$

Ex 3: Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

Then write $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ as a sum of two orthogonal vectors. Also, observe geometrically.

$$\hat{y} = \frac{-4 + 14}{16 + 4} \vec{u} = \frac{1}{2} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$



6.2: Orthogonal Sets

Ex 4: Find the distance from the vector $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ to the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ (from Ex 3).

\hat{y} is the closest point on the line through \vec{u} to \vec{y} .

$$\begin{aligned} \|\vec{z}\| &= \|\vec{y} - \hat{y}\| \\ &= \sqrt{9 + 36} \\ &= \sqrt{45} \end{aligned}$$

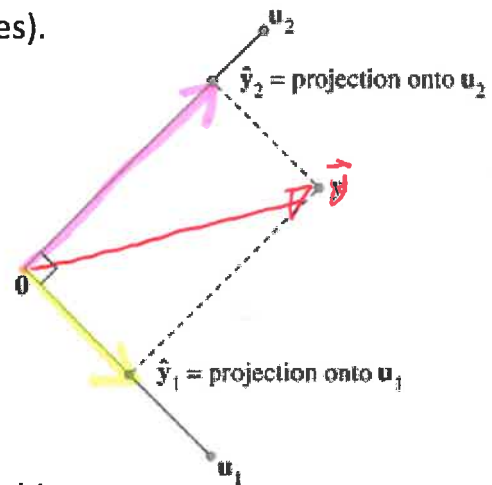
That is, we need $\|\vec{z}\|$

Notice that the orthogonal projection formula matches the weights of the orthogonal basis terms in theorem 5. Theorem 5 decomposes a vector into a sum of orthogonal projections onto one-dimensional subspaces (lines).

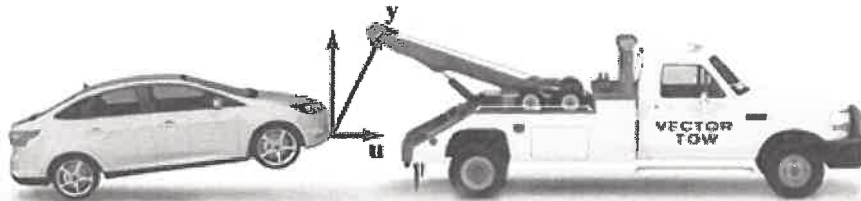
In \mathbb{R}^2 , if we have an orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2\}$

then any $\mathbf{y} \in \mathbb{R}^2$ can be written as

$$\vec{y} = \frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \mathbf{u}_1 + \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} \mathbf{u}_2$$



In Physics we use this to decompose force on an object.



A set of vectors $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is called an orthonormal set if it is an orthogonal set of unit vectors. If W is spanned by this set, then the set is an orthonormal basis for W .

The simplest orthonormal basis for \mathbb{R}^n is $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$

Any nonempty subset of this standard basis is orthonormal as well.

6.2: Orthogonal Sets

Ex 5: Determine whether the set of vectors is orthonormal. Is it an orthonormal basis for \mathbb{R}^3 ?

$$\vec{u}_1, \vec{u}_2, \vec{u}_3 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

verify the vectors are unit vectors.

$$\|\vec{u}_1\|^2 = \vec{u}_1 \cdot \vec{u}_1 = \frac{1}{10} + \frac{9}{20} + \frac{9}{20} = 1$$

$$\vec{u}_2 \cdot \vec{u}_2 = \frac{9}{10} + \frac{1}{20} + \frac{1}{20} = 1$$

$$\vec{u}_3 \cdot \vec{u}_3 = 0 + \frac{1}{2} + \frac{1}{2} = 1$$

They are unit vectors.

Theorem 6

\therefore they form an orthonormal basis for \mathbb{R}^3 .
An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

verify the vectors are orthonormal

$$\vec{u}_1 \cdot \vec{u}_2 = \frac{3}{10} - \frac{3}{20} - \frac{3}{20} = 0$$

$$\vec{u}_1 \cdot \vec{u}_3 = 0 - \frac{3}{\sqrt{40}} + \frac{3}{\sqrt{40}} = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = 0 + \frac{1}{\sqrt{40}} - \frac{1}{\sqrt{40}} = 0$$

They are orthogonal

Proof:

Let $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \dots & \vec{u}_p \end{bmatrix}$ where $\vec{u}_1, \dots, \vec{u}_p$ are orthonormal.

$$\begin{aligned} \Rightarrow U^T U &= \begin{bmatrix} \vec{u}_1^T \\ \vdots \\ \vec{u}_p^T \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \dots & \vec{u}_p \end{bmatrix} = \begin{bmatrix} \vec{u}_1^T \vec{u}_1 & \vec{u}_1^T \vec{u}_2 & \dots & \vec{u}_1^T \vec{u}_p \\ \vec{u}_2^T \vec{u}_1 & \vec{u}_2^T \vec{u}_2 & \dots & \vec{u}_2^T \vec{u}_p \\ \vdots & \vdots & \ddots & \vdots \\ \vec{u}_p^T \vec{u}_1 & \vec{u}_p^T \vec{u}_2 & \dots & \vec{u}_p^T \vec{u}_p \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \end{aligned}$$

Theorem 7 Q. E. D.

Let U be an $m \times n$ matrix with orthonormal columns, and let \mathbf{x} and \mathbf{y} be in \mathbb{R}^n . Then

orthogonal and unit vectors.

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$

6.2: Orthogonal Sets

Ex 6: Let $U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} \sqrt{6} \\ 4 \end{bmatrix}$. Verify that $\|U\mathbf{x}\| = \|\mathbf{x}\|$ LHS RHS

LHS

$$U\vec{x} = \begin{bmatrix} \frac{\sqrt{6}}{3} - \frac{4}{\sqrt{2}} \\ \frac{\sqrt{6}}{3} \\ \frac{\sqrt{6}}{3} + \frac{4}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 2 - 2\sqrt{2} \\ \sqrt{2} \\ 2 + 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \\ 3\sqrt{2} \end{bmatrix} \Rightarrow \|U\vec{x}\| = \sqrt{2+2+18} = \sqrt{22}$$

main point: length is preserved under an orthogonal transformation.

vector in \mathbb{R}^3

$\|\vec{x}\| = \sqrt{6+16} = \sqrt{22}$ ← The norms are equal.

vector in \mathbb{R}^2

An orthogonal matrix is a square invertible matrix U , such that $U^{-1} = U^T$. By theorem 6, it has orthonormal columns.

The matrix formed from the vectors from Ex 5 is an example.

$$\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} & 0 \\ 3/\sqrt{20} & -1/\sqrt{20} & -1/\sqrt{2} \\ 3/\sqrt{20} & -1/\sqrt{20} & 1/\sqrt{2} \end{bmatrix}$$

Practice Problem

1. Let U and \mathbf{x} be as in example 6, and let $\mathbf{y} = \begin{bmatrix} -\sqrt{3} \\ \sqrt{2} \end{bmatrix}$. Verify that $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

$$U\vec{y} = \begin{bmatrix} -1-1 \\ -1+0 \\ -1+1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} \Rightarrow (U\vec{x}) \cdot (U\vec{y}) = 2\sqrt{2} - \sqrt{2} + 0 = \sqrt{2}$$

And $\vec{x} \cdot \vec{y} = -3\sqrt{2} + 4\sqrt{2} = \sqrt{2}$ ← the inner products are equal

main point: angle is preserved under an orthogonal transformation.