

5.3: Diagonalization

Math 220: Linear Algebra

Ex 1: If $D = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ find D^2, D^3 , and D^k .

$$D^2 = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & 4^2 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 3^3 & 0 \\ 0 & 4^3 \end{bmatrix}$$

$$D^k = \begin{bmatrix} 3^k & 0 \\ 0 & 4^k \end{bmatrix}$$

If $A = PDP^{-1}$ for some invertible P and diagonal D , then A^k is also easy to compute.

Ex 2: Let $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$. Find a formula for A^k given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$P^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 2 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

explore: $A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$

$$\text{and } A^k = PD^kP^{-1}$$

$$\Rightarrow A^k = \begin{bmatrix} -2 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 5^k \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{2} \end{bmatrix}$$

so where did P and D come from?

eigenvalues: solve $0 = \begin{vmatrix} 7-\lambda & 4 \\ -3 & -1-\lambda \end{vmatrix}$

$$= (7-\lambda)(-1-\lambda) + 12$$

$$= \lambda^2 - 6\lambda + 5$$

$$= (\lambda - 5)(\lambda - 1)$$

so $\lambda = 5$ or $\lambda = 1$

eigen vectors: $\begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

A square matrix is said to be diagonalizable if A is similar to a diagonal matrix.

Theorem 5 The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A . In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P .

These eigenvectors, since they are linearly independent, form a basis of \mathbb{R}^n .

Ex 3: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 3 & -1 \\ -1 & -2 & 2 \end{bmatrix}$. That is, find an invertible

matrix P and diagonal matrix D such that $A = PDP^{-1}$. The eigenvalues are $\lambda = 1, 5$.

we need the eigenvectors.

$\lambda = 1$

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ \& } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

These are the eigenvectors that go w/ $\lambda = 1$.

$\lambda = 1$ has multiplicity 2.

$\lambda = 5$

$$\begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

This is the eigenvector associated w/ $\lambda = 5$

$\lambda = 5$ has multiplicity 1

$$\Rightarrow A = PDP^{-1} \text{ w/ } P = \begin{bmatrix} -2 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Invertible matrix whose columns form an eigenbasis for \mathbb{R}^n

Diagonal matrix whose entries are eigenvalues.

Ex 4:

Diagonalize the matrix, if possible.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

The eigenvalues are $\lambda = 4$ (algebraic) mult: 2 and $\lambda = 5$

$\lambda = 4$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ This is the eigenvector associated w/ $\lambda = 4$.

$\lambda = 4$ has (geometric) multiplicity 1.

$\lambda = 5$ This will have one eigenvector.

There aren't enough eigenvectors to diagonalize P
 $\therefore A$ is not diagonalizable.

Theorem 6

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

This is

Not a requirement though for diagonalizable though, as we saw in Ex 3.

Theorem 7

Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \dots, \lambda_p$.

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k .

b. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n , and this happens if and only if (i) the characteristic polynomial factors completely into linear factors and (ii) the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k . No complex eigenvalues

c. If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k , then the total collection of vectors in the sets B_1, \dots, B_p forms an eigenvector basis for \mathbb{R}^n .

see example 4 above

Ex 5: Diagonalize the matrix, if possible. $A = \begin{bmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

Our eigenvalues are $\lambda = 2, 3, 5$

$\lambda = 2$ (Try this 1st as the only way A will not be diagonalizable is if the (geometric) multiplicity is 1.

$$\begin{bmatrix} 3 & -3 & 0 & 9 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

multiplicity 2

$\lambda = 3$

$$\begin{bmatrix} 2 & -3 & 0 & 9 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda = 5$

$$\begin{bmatrix} 0 & -3 & 0 & 9 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$A = PDP^{-1} \text{ w/ } P = \begin{bmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

and thus A is diagonalizable.

Practice Problems

1. Compute A^8 , where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that \mathbf{v}_1 and \mathbf{v}_2 are eigenvectors of A . Use this information to diagonalize A .

3. Let A be a 4×4 matrix with eigenvalues 5, 3, and -2 , and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?

powers of A tell us to diagonalize

5.3: Diagonalization

Practice Problems

1. Compute A^8 , where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

eigenvalues

$$\text{solve } 0 = \begin{vmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) + 6 = \lambda^2 - 3\lambda + 2$$

$$\Rightarrow 0 = (\lambda-2)(\lambda-1) \text{ or } \lambda = 1, 2$$

eigenvectors

$\lambda=1$: $\begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector associated with $\lambda=1$.

$\lambda=2$: $\begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ eigenvectors associated w/ $\lambda=2$.

$A = P D P^{-1}$ where $P = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ so $A^8 = P D^8 P^{-1} = P \begin{bmatrix} 1^8 & 0 \\ 0 & 2^8 \end{bmatrix} P^{-1}$

$$= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 256 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are

told that \mathbf{v}_1 and \mathbf{v}_2 are eigenvectors of A . Use this information to diagonalize A .

we need the eigenvalues.

$$\begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ AND } \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so the eigenvalues are $\lambda=1, 3$ and $A = P D P^{-1}$ w/ $P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

3. Let A be a 4×4 matrix with eigenvalues 5, 3, and -2 , and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?

Yes. $\underbrace{\dim(E_{\lambda=3})}_{2} + \underbrace{\dim(E_{\lambda=5})}_{\geq 1} + \underbrace{\dim(E_{\lambda=-2})}_{\geq 1} \geq 4$

but the max it can be is 4

There are enough LI eigenvectors to diagonalize.