4.3: Linearly Independent Sets; Bases

Math 220: Linear Algebra

Recall the previous definitions of Linearly Independent and Linearly Dependent. We are now going to think in terms of a Vector Space V, rather than just $\mathbb{R}^n$.

Definition
An indexed set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\cancel{\mathbb{R}^n}$ V is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

And recall that

Theorem 4
An indexed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

the easily solved matrix equation

If a vector space is not ~~just an~~ $\mathbb{R}^n$ with ~~an easy~~ $A\mathbf{x} = \mathbf{0}$, then we need Theorem 4 to show a linear dependence relation to prove linear dependence.

Ex 1: Discuss the linear dependence or independence of the following sets on $C[0,1]$, the space of all continuous functions on $0 \le t \le 1$.

$\{\sin t, \cos t\}$

cos(t) = a sin(t)
this is not true for all $t \in [0,1]$ so sin(t) and cos(t) are linearly independent.

$\{\sin t \cos t, \sin 2t\}$

sin 2t = a sin(t)cos(t)
true when a = 2
so sin(t)cos(t) and sin(2t) are linearly dependent.

4.3: Linearly Independent Sets; Bases

Definition
Let H be a subspace of a vector space V. An indexed set of vectors $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

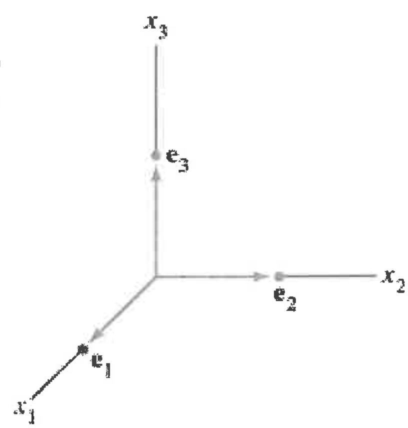
(i) B is a linearly independent set, and

(ii) the subspace spanned by B coincides with H; that is,

$$H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

Ex 2: What can we say about an invertible matrix A? $n \times n$

(i) cols of A are Lin. Ind.
(ii) cols of A span $\mathbb{R}^n$
$\therefore$ columns of A are a basis for $\mathbb{R}^n$.



The columns of the identity matrix, $\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n$ is called the standard basis for $\mathbb{R}^n$.

Ex 3: Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis for $\mathbb{R}^3$.

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

This is not a basis.

$$\text{rref}\left(\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}\right) \sim \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

free variable
so not linearly independent,
$\vec{v}_3 = \frac{1}{2}\vec{v}_1 + 2\vec{v}_2$

4.3: Linearly Independent Sets; Bases

Do $\{\mathbf{v}_1, \mathbf{v}_2\}$ form a basis for $\mathbb{R}^2$? No, since $\vec{v}_1, \vec{v}_2 \notin \mathbb{R}^2$

Ex 4: Let $S = \{1, t, t^2, \ldots, t^n\}$. Verify that S is a basis for $\mathbb{P}_n$. This basis is called the **standard basis** for $\mathbb{P}_n$.

(i) $a_0 \cdot 1 + a_1 t + a_2 t^2 + \cdots + a_n t^n = 0$ for all t

$\Rightarrow a_0 = a_1 = \cdots = a_n = 0$

(ii) S clearly spans $\mathbb{P}_n$

$\therefore$ S is a basis for $\mathbb{P}_n$

A basis is an "efficient" spanning set because it contains no unnecessary vectors.

Ex 5: Let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ as in Ex 3. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$$

we already showed that

$\vec{v}_3 = \frac{1}{2}\vec{v}_1 + 2\vec{v}_2$

$\Rightarrow \vec{v}_3 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$

$\therefore \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \text{span}\{\vec{v}_1, \vec{v}_2\}$

Theorem 5 The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\ \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

Proof:

see next page

proof.

Reorder/reindex $\vec{v}_1, \dots, \vec{v}_p$ so (a) $\vec{v}_k$ is now $\vec{v}_p$, and (b) the vectors are linearly independent until they become linearly dependent.

$$\underbrace{\vec{v}_1, \dots, \vec{v}_\ell}_{\text{L.I.}}, \underbrace{\vec{v}_{\ell+1}, \dots, \vec{v}_p}_{\substack{\text{L.D. w/ } v_1, \dots, v_\ell \\ \text{and } \vec{v}_p \text{ is the} \\ \text{old } \vec{v}_k}}$$

(a) Let $\vec{x} \in H$ be given. There exist $c_1, \dots, c_p$ s.t.

$$\vec{x} = c_1 \vec{v}_1 + \dots + c_{p-1} \vec{v}_{p-1}\ c_p \vec{v}_p$$

and $\vec{v}_p = d_1 \vec{v}_1 + \dots + d_{p-1} \vec{v}_{p-1}$

$$\Rightarrow \vec{x} = c_1 \vec{v}_1 + \dots + c_{p-1} \vec{v}_{p-1} + c_p \overbrace{(d_1 \vec{v}_1 + \dots + d_{p-1} \vec{v}_{p-1})}^{\vec{v}_p}$$

$$= (c_1 + c_p d_1) \vec{v}_1 + \dots + (c_{p-1} + c_p d_{p-1}) \vec{v}_{p-1}$$

$\therefore$ The set formed by removing the old $\vec{v}_k$ still spans H.

(b) By construction, the reordered $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent until they aren't. Thus the subset $\vec{v}_1, \dots, \vec{v}_\ell$ is L.I. and spans H.

$\therefore$ $\{\vec{v}_1, \dots, \vec{v}_\ell\}$ are a basis for H.

We already know how to find a basis for the Nul A, as we saw that the row reduced system that describes the solutions of Nul A, is already linearly independent.

However, finding a basis for Col A that doesn't have unneeded vectors is our next step.

Ex 6: Find a Basis for Col B where

$$B=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & b_4 & \mathbf{b}_4 & \mathbf{b}_5\end{bmatrix}=\begin{bmatrix}1 & 0 & -3 & 0 & 4\\ 0 & 1 & -4 & 0 & -5\\ 0 & 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

basis for Col B = $\{\vec{b}_1, \vec{b}_2, \vec{b}_4\}$

we exclude $\vec{b}_3$ and $\vec{b}_5$ because $\vec{b}_3 = -3\vec{b}_1 - 4\vec{b}_2$

and $b_5 = 4\vec{b}_1 - 5\vec{b}_2 - 2\vec{b}_4$

Ex 7: Find a Basis for Col A where, A reduces to the matrix B in the previous example.

$$A=\begin{bmatrix}1 & 0 & -3 & 1 & 2\\ 0 & 1 & -4 & -3 & 1\\ -3 & 2 & 1 & -8 & -6\\ 2 & -3 & 6 & 7 & 9\end{bmatrix}$$

basis for col $A = \left\{ \begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\-3\\-8\\7\end{bmatrix} \right\}$

Since $A\mathbf{x}=\mathbf{0}$ and the reduced echelon form $B\mathbf{x}=\mathbf{0}$ have the exact same solution sets, then their columns have the exact same dependence relationships. Let's check.

$$-3\begin{bmatrix}1\\0\\-3\\2\end{bmatrix} - 4\begin{bmatrix}0\\1\\2\\-3\end{bmatrix} = \begin{bmatrix}-3\\-4\\1\\6\end{bmatrix}$$

WARNING:

You must use the original pivot columns of A. Why doesn't $\text{Col}A = \text{Span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_4\}$?

pivot columns ⇒ tell us which columns to use from A to make a basis for col A.

Theorem 6
The pivot columns of a matrix A form a basis for Col A.

$$\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\} \neq \text{col } A = \text{span}\left\{\begin{bmatrix}1\\0\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\1\\2\\-3\end{bmatrix}, \begin{bmatrix}1\\-3\\-8\\7\end{bmatrix}\right\}$$

A Basis is basically the smallest spanning set possible. Remove any vectors from it, and the set is no longer spanned, add any vectors to it, and it becomes linearly dependent.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\3\\0\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Linearly independent but does not span $\mathbb{R}^3$	A basis for $\mathbb{R}^3$	Spans $\mathbb{R}^3$ but is linearly dependent
↑	↑	↑
Too few	Enough to span And Lin. Ind.	Too many.

Practice Problems

1. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-2\\7\\-9\end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$.
Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

$\{\vec{v}_1, \vec{v}_2\}$ are L. I. but do not span. thus they are not a basis for $\mathbb{R}^3$.

$\vec{v}_1, \vec{v}_2 \notin \mathbb{R}^2$ so aren't a basis for $\mathbb{R}^2$.

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix} -4 \\ -8 \\ 9 \end{bmatrix}$. Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

$$\text{rref}\left(\begin{array}{c} \begin{bmatrix} 1 & 6 & 2 & -4 \\ -3 & 2 & -2 & -8 \\ 4 & -1 & 3 & 9 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & & \\ \text{col } 1 & \text{col } 2 & & \end{array} \end{array} \right) \sim \begin{array}{c} \begin{bmatrix} 1 & 0 & 4/5 & 2 \\ 0 & 1 & 1/5 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & & \\ \text{col } 1 & \text{col } 2 & & \end{array} \end{array}$$

$$\text{Basis for } W = \left\{ \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 6 \\ 2 \\ -1 \end{bmatrix} \right\}$$

3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $H = \left\{ \begin{bmatrix} s \\ s \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$$\begin{bmatrix} s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H?

no $\vec{v}_1, \vec{v}_2 \notin H$. (Too much is included in span $\{v_1, v_2\}$).