1.9: Matrix of a Linear Transformation

Math 220: Linear Algebra

Ex 1: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2 \to \mathbb{R}^3$ such that $T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix}$. ← column vectors

Find a formula for the image of an arbitrary $\mathbf{x} \in \mathbb{R}^2$. ← $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2)$$

$$= x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= x_1 \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -1 \\ 9 \end{bmatrix} \quad \text{(linear combination)}$$

$$= \begin{bmatrix} 3 & 0 \\ 2 & -1 \\ -5 & 9 \end{bmatrix} \vec{x}$$

This shows us that knowing $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ can give us $T(\mathbf{x})$ for any $\mathbf{x}$. in $\mathbb{R}^2$

That is: $T(\mathbf{x}) = \begin{bmatrix} | & | \\ T(\mathbf{e}_1) & T(\mathbf{e}_2) \\ | & | \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = A\mathbf{x} \quad \forall \vec{x} \in \mathbb{R}^2$

col vecs

Theorem 10
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the identity matrix in $\mathbb{R}^n$:

$$A = [T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)] \qquad (3)$$

This Matrix A is called the standard matrix for the linear transformation.

Ex 2: Find the standard matrix A for the contraction transformation $T(\mathbf{x})=\frac{1}{2}\mathbf{x}$ for $\mathbf{x}\in\mathbb{R}^2$.

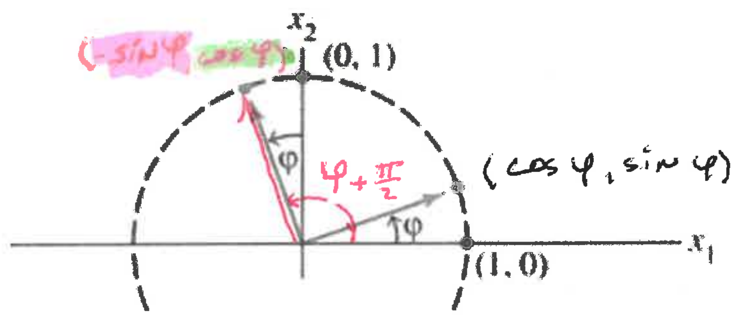
$\underline{1st}$: $T(\vec{e}_1) = \frac{1}{2}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1/2\\0\end{bmatrix}$

$T(\vec{e}_2) = \frac{1}{2}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1/2\end{bmatrix}$

$\underline{2nd}$: $A = \begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

Ex 3: Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through the angle φ, with counterclockwise rotation for a positive angle (see the figure). Find the standard matrix A of this transformation. φ: Greek letter phi

$\underline{1st}$: $T(\vec{e}_1) = \begin{bmatrix}\cos\varphi\\ \sin\varphi\end{bmatrix}$



$T(e_2) = \begin{bmatrix}\cos(\varphi+\frac{\pi}{2})\\ \sin(\varphi+\frac{\pi}{2})\end{bmatrix} = \begin{bmatrix}-\sin\varphi\\ \cos\varphi\end{bmatrix}$

recall

$\cos(\varphi+\frac{\pi}{2}) = \cos\varphi\overbrace{\cos\frac{\pi}{2}}^{0} - \sin\varphi\overbrace{\sin\frac{\pi}{2}}^{1}$

$= -\sin\varphi$

$\sin(\varphi+\frac{\pi}{2}) = \sin\varphi\underbrace{\cos\frac{\pi}{2}}_{0} + \underbrace{\sin\frac{\pi}{2}}_{1}\cos\varphi$

$= \cos\varphi$

$\underline{2nd}$: $A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$

Applications of Linear Transformations

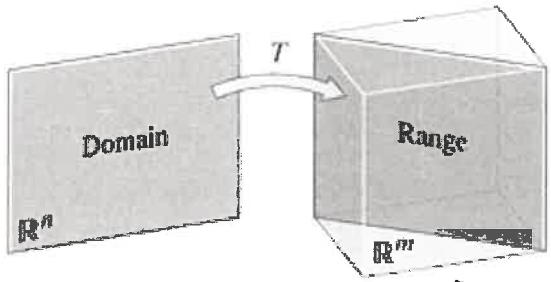
Ex 4: Observe and discuss in the interactive ebook: *(also, pages 74-76)*

- Reflection $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ x_1-axis, $\begin{bmatrix}-1&0\\0&1\end{bmatrix}$ x_2-axis, $\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $x_1=x_2$, $\begin{bmatrix}0&-1\\-1&0\end{bmatrix}$ $x_1=-x_2$, $\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$ origin
- Contraction & Expansion $\begin{bmatrix}b&0\\0&1\end{bmatrix}$ x_1 expansion, $\begin{bmatrix}1&0\\0&b\end{bmatrix}$ x_2 expansion
- Shear $\begin{bmatrix}1&b\\0&1\end{bmatrix}$ parallel to x_1-axis, $\begin{bmatrix}1&0\\b&1\end{bmatrix}$ parallel to x_2 axis
- Projection (in $\mathbb{R}^3$) $\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ onto the x_1,x_2 plane, $\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$ onto the x_1,x_3 plane

Theory of Linear Transformations

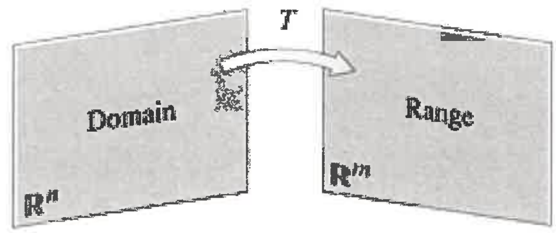
Definition
A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be <u>onto</u> $\mathbb{R}^m$ if each **b** in $\mathbb{R}^m$ is the image of *at least one* **x** in $\mathbb{R}^n$.

Another way of saying this is that the range of *T* is all of the codomain $\mathbb{R}^m$



T is *not* onto $\mathbb{R}^m$

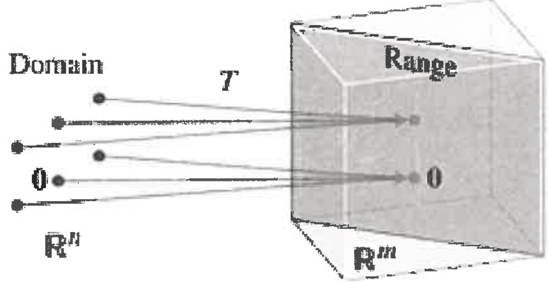
range does not fill the codomain



T is onto $\mathbb{R}^m$

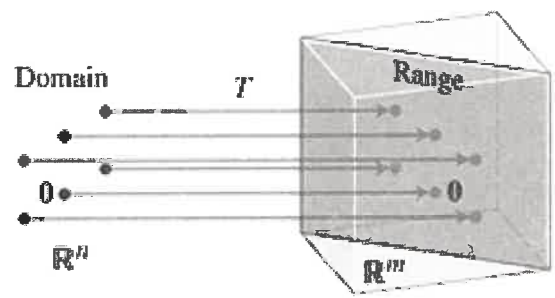
range fills the codomain

Definition
A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** if each **b** in $\mathbb{R}^m$ is the image of *at most one* **x** in $\mathbb{R}^n$.



T is *not* one-to-one

In precalc this meant f failed the vertical line test



T is one-to-one

In precalc this meant f passed the vertical line test

Theorem 11
<u>claim</u>; Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then *T* is one-to-one if and only if the equation $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution.

<u>**Proof**</u>, ($\Rightarrow$) Assume T is one-to-one.
we know $T(\vec{0}) = \vec{0}$ (since T is a L.T.)
$\Rightarrow T(\vec{x}) = \vec{0}$ has a solution and it must be unique since T is one-to-one.

($\Leftarrow$) Assume $T(\vec{x}) = \vec{0}$ has only the trivial solution.
Let's suppose T is not one-to-one.
$\Rightarrow \exists\ \vec{u} \neq \vec{v}$ and $\vec{b}$ s.t. $T(\vec{u}) = \vec{b}$ and $T(\vec{v}) = \vec{b}$
$\Rightarrow T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v}) = \vec{b} - \vec{b} = \vec{0}$
$\Rightarrow \vec{u} = \vec{v}$ $\Rightarrow\Leftarrow$ contradiction
$\Rightarrow T$ is one-to-one
$\therefore T$ is one-to-one iff $T(\vec{x}) = \vec{0}$ has only the trivial solution

Theorem 12

claim: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$;

b. T is one-to-one if and only if the columns of A are linearly independent.

Proof.

(a) Assume the columns of A span $\mathbb{R}^m$.

$\Leftrightarrow$ There is a solution to $A\vec{x} = \vec{b}$ for all $\vec{b} \in \mathbb{R}^m$

$\Leftrightarrow$ There is a solution to $T(\vec{x}) = \vec{b}$ for all $b \in \mathbb{R}^m$

$\Leftrightarrow$ T maps onto $\mathbb{R}^m$.

(b) Assume T is 1-to-1

$\Leftrightarrow$ $T(\vec{x}) = 0$ has only the trivial solution

$\Leftrightarrow$ $A\vec{x} = 0$ has only the trivial solu

$\Leftrightarrow$ columns of A are L.I.

Q.E.D.

Ex 5: Let T be the linear transformation whose standard matrix is below (2 cases). Determine whether they are "onto $\mathbb{R}^3$" and/or a one-to-one mapping.

a) $A = \begin{bmatrix} 1 & -2 & 3 & 1 \\ 0 & 0 & 2 & -5 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & -2 \\ 2 & -4 \\ 3 & 5 \end{bmatrix}$

	Why?	Why?
onto $\mathbb{R}^3$?	Yes, the columns span $\mathbb{R}^3$ since there are 3 pivots.	No, not enough vectors to span $\mathbb{R}^3$
one-to-one?	No, too many column vectors.	Yes, the columns are L.I.