

# 1.7: Linear Independence

## Math 220: Linear Algebra

### Definition

An indexed set of vectors  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is said to be **linearly independent** if the vector equation

$$x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + \dots + x_p \mathbf{v}_p = \mathbf{0}$$

has only the **trivial solution**. The set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  is said to be **linearly dependent** if there exist weights  $c_1, \dots, c_p$ , not all zero, such that

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_p \mathbf{v}_p = \mathbf{0}$$

**Ex 1:** Determine whether the set  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is linearly independent. If not, find a linear dependence relation among  $\mathbf{v}_1, \mathbf{v}_2$ , and  $\mathbf{v}_3$ .

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{ and } \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2x_3 \\ x_2 &= -x_3 \\ x_3 &= x_3 \text{ (free)} \end{aligned} \Rightarrow \vec{x} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

so, for example, if  $x_3 = 1$  this

tells us  $2\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$ . This is a non-trivial solution to the homogeneous equation so  $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$  are linearly dependent.

## 1.7: Linear Independence

The columns of a matrix  $A$  are linearly independent if and only if the equation  $Ax = \mathbf{0}$  has *only* the trivial solution.

This is a big deal!

Ex 2: Determine whether the columns of the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -1 \\ -3 & 1 & -2 \end{bmatrix}$  are linearly independent.

$$\Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & 1 & -1 & 0 \\ -3 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \left. \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array} \right\} \text{the trivial solution is the only solution so the columns of } A \text{ are linearly independent.}$$

Ex 3: What about?

Linearly Dependent or Independent? Why?  
(L.D.) (L.I.)

$\{\mathbf{v}\}$ , not the zero vector L.I. since  $c\mathbf{v} = \mathbf{0} \Rightarrow c = 0$

$\{\mathbf{0}\}$  L.D. since  $c \cdot \mathbf{0} = \mathbf{0}$  for  $c \in \mathbb{R}$

$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix} \right\}$  L.D.  $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$   
that is  $3 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$  L.I.  $\begin{bmatrix} 1 & -3 & 0 \\ -2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$   
only the trivial solution.

## 1.7: Linear Independence

A set of two vectors  $\{\mathbf{v}_1, \mathbf{v}_2\}$  is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

### Theorem 7 Characterization of Linearly Dependent Sets

claim: An indexed set  $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  of two or more vectors is linearly dependent if and only if at least one of the vectors in  $S$  is a linear combination of the others. In fact, if  $S$  is linearly dependent and  $\mathbf{v}_1 \neq \mathbf{0}$ , then some  $\mathbf{v}_j$  (with  $j > 1$ ) is a linear combination of the preceding vectors,  $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$ .

**Proof:**

$(\Rightarrow)$  Assume  $S$  is linearly dependent.

If  $\vec{v}_1 = \vec{0}$  then it is a linear combination of the other vectors:  $0\vec{v}_2 + \dots + 0\vec{v}_p = \vec{v}_1$ .

If  $\vec{v}_1 \neq \vec{0}$  then  $c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1} + c_j\vec{v}_j + c_{j+1}\vec{v}_{j+1} + \dots + c_p\vec{v}_p = \vec{0}$  where not all of  $c_1, \dots, c_p$  are zero. Suppose  $c_j \neq 0$  and  $c_{j+1} = \dots = c_p = 0$ .

$$\Rightarrow c_1\vec{v}_1 + \dots + c_j\vec{v}_j + 0\vec{v}_{j+1} + \dots + 0\vec{v}_p = \vec{0}$$

$$\Rightarrow \vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \dots - \frac{c_{j-1}}{c_j}\vec{v}_{j-1}$$

so a vector is a linear combination of the others.

$(\Leftarrow)$  Suppose one vector, call it  $\vec{v}_j$ , is a linear combination of the others. (Note: we can always reorder if need be to make this true)

$$\Rightarrow \exists c_1, \dots, c_{j-1} \in \mathbb{R} \text{ s.t. } \vec{v}_j = c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1}$$

$$\Rightarrow c_1\vec{v}_1 + \dots + c_{j-1}\vec{v}_{j-1} - \vec{v}_j + 0\vec{v}_{j+1} + \dots + 0\vec{v}_p = \vec{0}$$

↑  
not zero

$\Rightarrow$  there is a non-trivial solution to the homogeneous equation and  $S$  is linearly dependent.

$\therefore S$  is L.D. iff one or more vectors in  $S$  is a linear combination of the others

## 1.7: Linear Independence

claim:

**Ex 4:** Given the set of vectors  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \in \mathbb{R}^3$  with  $\mathbf{u}$  and  $\mathbf{v}$  linearly independent, explain why vector  $\mathbf{w}$  is in the plane spanned by  $\mathbf{u}$  and  $\mathbf{v}$  if and only if  $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$  is linearly dependent.

proof.

Suppose  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$  w/  $\vec{u}, \vec{v}$  L.I. and  $w \in \text{Span}\{\vec{u}, \vec{v}\}$

$$\Leftrightarrow w = a\vec{u} + b\vec{v}$$

$\Leftrightarrow \vec{0} = a\vec{u} + b\vec{v} - 1\vec{w}$  which is a non-trivial solution to the homogeneous equation and thus  $\{\vec{u}, \vec{v}, \vec{w}\}$  is linearly dependent.

Q.E.D.

### Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set  $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$  in  $\mathbb{R}^n$  is linearly dependent if  $p > n$ .

Proof:

Let  $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$  w/  $p > n$  be given.

less rows

$$\begin{bmatrix} | & & | \\ \vec{v}_1 & \dots & \vec{v}_p \\ | & & | \end{bmatrix}_{n \times p} \sim \begin{bmatrix} \text{at most} \\ n \\ \text{pivots} \end{bmatrix}$$

less columns

The main point is that too many vectors guarantees linear dependence.

$\Rightarrow$  free variables

$\Rightarrow A\vec{x} = \vec{0}$  has non-trivial solutions

$\therefore \vec{v}_1, \dots, \vec{v}_p$  are L.D.

**Ex 5:** Using Theorem 8, create a set of vectors in  $\mathbb{R}^3$  that is linearly dependent, and don't automatically make some of the vectors obvious multiples or combinations of the others.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Any 4+ vectors in  $\mathbb{R}^3$  do the trick.

## 1.7: Linear Independence

### Theorem 9

If a set  $S = \{v_1, \dots, v_p\}$  in  $\mathbb{R}^n$  contains the zero vector, then the set is linearly dependent.

Proof:

Let  $\{v_1, \dots, v_p\}$  in  $\mathbb{R}^n$  contain  $\vec{0}$ .

WLOG assume  $\vec{v}_1 = \vec{0}$

$$\Rightarrow 1\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_p = \vec{0}$$

$\Rightarrow$  there is a non-trivial solution to the homogeneous equation.

$\therefore$  the vectors are L.D.

Ex 6: Determine by inspection if the give set is linearly dependent.

a.  $\begin{bmatrix} 1 \\ 7 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$  L.D. (too many vecs)

b.  $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$  L.D. (includes  $\vec{0}$ )

c.  $\begin{bmatrix} -2 \\ 4 \\ 6 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \\ -9 \\ 15 \end{bmatrix}$  L.I. (Not scalar multiples)

## 1.7: Linear Independence

**Ex 7:** Network flow exercise from 1.6 (we did a chemistry example previously).

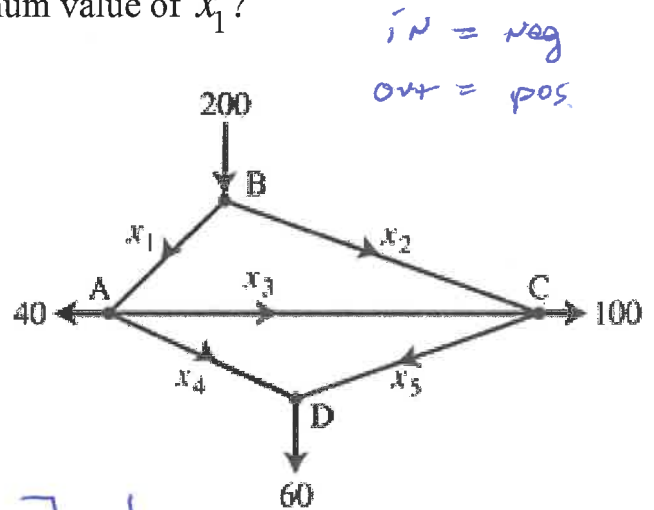
- Find the general traffic pattern in the freeway network shown in the figure. (Flow rates are in cars/minute)
- Describe the general traffic pattern when the road whose flow is  $x_4$  is closed.
- When  $x_4 = 0$ , what is the minimum value of  $x_1$ ?

$$A: -x_1 + x_3 + x_4 + 40 = 0$$

$$B: x_1 + x_2 - 200 = 0$$

$$C: -x_2 - x_3 + x_5 + 100 = 0$$

$$D: -x_4 - x_5 + 60 = 0$$



$$\Rightarrow \begin{bmatrix} -1 & 0 & 1 & 1 & 0 & -40 \\ 1 & 1 & 0 & 0 & 0 & 200 \\ 0 & -1 & -1 & 0 & 1 & -100 \\ 0 & 0 & 0 & -1 & -1 & -60 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 0 & -1 & 0 & 1 & 100 \\ 0 & \boxed{1} & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & \boxed{1} & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) If  $x_4 = 0$

$$x_1 = 40 + x_3$$

$$x_2 = 160 - x_3$$

$$x_3 = x_3 \text{ (free)}$$

$$x_4 = 0$$

$$x_5 = 60$$

(a)

$$\Rightarrow \begin{aligned} x_1 &= 100 + x_3 - x_5 \\ x_2 &= 100 - x_3 + x_5 \\ x_3 &= x_3 \text{ (free)} \\ x_4 &= 60 - x_5 \\ x_5 &= x_5 \text{ (free)} \end{aligned}$$

(c) since  $x_3 \geq 0$   
the minimum  
of  $x_1 = 40$ .