1.1: Systems of Linear Equations
Math 220: Linear Algebra

Linear Equations

A ___linear___ ___equation___ of the variables $x_1, x_2, ..., x_n$ has the form

$$a_1 x_1 + a_2 x_2 + ... + a_n x_n = b$$

Where b, a_1, a_2, ..., a_n are real or complex numbers.

Ex 1: Circle the linear equations, and state why the non-linear aren't linear.

(a) $3x_1 - x_2 = 5x_1$ b) $4x_2 + 5 = \sqrt{x_1}$

no multiplication of variables ←   ↖ no roots

c) $x_1 - 4x_2 = x_1 x_2$ (d) $-2x_1 + 7x_2 - \pi x_3 = \sqrt{2}$

A ___system___ of ___linear___ ___equations___ is a collection of one or more linear equations with the same variables. For example

$$\begin{aligned} 3x_1 - x_2 - 4x_3 &= 3 \\ x_1 \qquad -5x_3 &= -2 \end{aligned}$$

A ___solution___ of a system is a list of numbers $\left(s_1, s_2, ..., s_n\right)$ that make every equation of the system true, when each s_k is substituted for x_k.

Ex 2: Verify that (3,2,1) is a solution to the system $\begin{aligned} 3x_1 - x_2 - 4x_3 &= 3 \\ x_1 \qquad -5x_3 &= -2 \end{aligned}$.
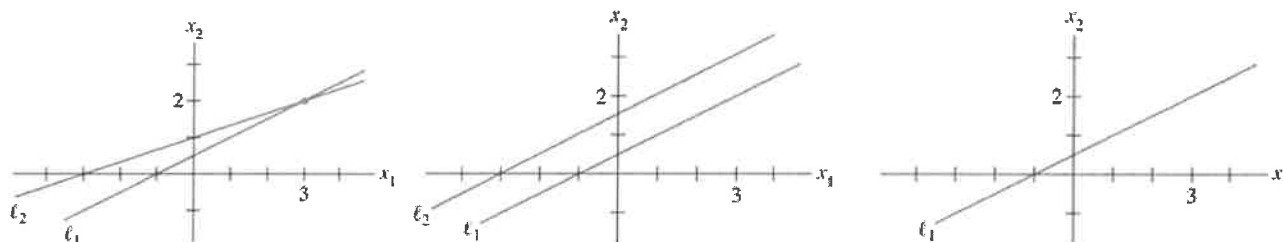
The set of all possible solutions is called the ___solution___ ___set___.

Ex 3: Find another solution to the system from Ex 2.

1.1: Systems of Linear Equations

Two systems are considered ___equivalent___ if they have the same solution set.

From 2 dimensional systems of equations in algebra, we should remember that there are 3 possibilities for the number of solutions to a system.

A system of linear equations has

1. no solution, or

2. exactly one solution, or

3. infinitely many solutions.

A system is called ___consistent___ if it has at least one solution, and ___inconsistent___ if it has no solutions.

Matrix Notation

We will represent a system of equations by its coefficients in a ___matrix___.

$$x_1 \qquad -3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2$$

will be re-written as the

___coefficient___ matrix

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix} \text{ 3x3}$$

The ___size___ of a matrix tells how many ___rows___ and ___columns___ a matrix has.

An $m \times n$ matrix has

___m___ ___rows___ and
___n___ ___columns___.

___augmented___ matrix

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \text{ 3x4}$$

1.1: Systems of Linear Equations

Solving a Linear System – We are going to describe an algorithm for solving linear systems, which replaces one system with an equivalent one that is easier to solve. Since they are equivalent, they have the same solution set.

Ex 4: Solve the system

Three Operations we can use:

- replacement
- interchange
- scaling

$$x_1 \qquad -3x_3 = 8$$
$$2x_1 + 2x_2 + 9x_3 = 7$$
$$x_2 + 5x_3 = -2$$

write as an augmented matrix. ↓

Step: $-2E_1 + E_2 \mapsto E_2$
Replacement
$$x_1 \qquad -3x_3 = 8$$
$$2x_2 + 15x_3 = -9$$
$$x_2 + 5x_3 = -2$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 2 & 2 & 9 & 7 \\ 0 & 1 & 5 & -2 \end{bmatrix} \quad -2R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 2 & 15 & -9 \\ 0 & 1 & 5 & -2 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

Step: $E_2 \leftrightarrow E_3$
Interchange
$$x_1 \qquad -3x_3 = 8$$
$$x_2 + 5x_3 = -2$$
$$2x_2 + 15x_3 = -9$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 2 & 15 & -9 \end{bmatrix} \quad -2R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 5 & -5 \end{bmatrix} \quad \tfrac{1}{5} R_3 \to R_3$$

Step: $-2E_2 + E_3 \to E_3$
Replacement
$$x_1 \qquad -3x_3 = 8$$
$$x_2 + 5x_3 = -2$$
$$5x_3 = -5$$

$$\begin{bmatrix} 1 & 0 & -3 & 8 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_1 + 3R_3 \to R_1 \\ R_2 - 5R_3 \to R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{which means}$$

Step: $\tfrac{1}{5} E_3 \to E_3$
scale
$$x_1 \qquad -3x_3 = 8$$
$$x_2 + 5x_3 = -2$$
$$x_3 = -1$$

Step: $E_1 + 3E_3 \to E_1$
$$E_2 - 5E_3 \to E_2$$
Replacement
$$x_1 \qquad = 5$$
$$x_2 \qquad = 3$$
$$x_3 = -1$$

so $(x_1, x_2, x_3) = (5, 3, -1)$

Elementary Row Operations

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.

2. (Interchange) Interchange two rows.

3. (Scaling) Multiply all entries in a row by a nonzero constant.

Two matrices are called ___row___ ___equivalent___ if there are a sequence of elementary row operations that transform one matrix into the other.

If two systems are row equivalent, they have the same ___solution___ ___set.___

Two Fundamental Questions About a Linear System

1. Is the system consistent; that is, does at least one solution *exist*?

2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

Ex 5: Determine whether the systems are consistent or inconsistent. Do not solve.

$$x_2 + 4x_3 = -5$$
$$x_1 + 3x_2 + 5x_3 = -2$$
$$3x_1 + 7x_2 + 7x_3 = 6$$

$$\begin{bmatrix} 0 & 1 & 4 & -5 \\ 1 & 3 & 5 & -2 \\ 3 & 7 & 7 & 6 \end{bmatrix} R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 3 & 7 & 7 & 6 \end{bmatrix} R_3 - 3R_1 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & -2 & -8 & 12 \end{bmatrix} R_3 + 2R_2 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 3 & 5 & -2 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

This means $0 = 2$

The system is inconsistent

$$x_1 + 3x_3 = 2$$
$$x_2 - 3x_4 = 3$$
$$-2x_2 + 3x_3 + 2x_4 = 1$$
$$3x_1 + 7x_4 = -5$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 3 & 0 & 0 & 7 & -5 \end{bmatrix} R_4 - 3R_1 \rightarrow R_4$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & -3 & 3 \\ 0 & -2 & 3 & 2 & 1 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} R_2 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & -3 & 3 \\ 0 & 0 & 3 & -1 & 4 \\ 0 & 0 & -9 & 7 & -11 \end{bmatrix} R_4 + 3R_3 \rightarrow R_4$$

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 2 & 0 & -3 & 3 \\ 0 & 0 & 3 & -1 & 4 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix}$$

There will be values for $x_1, \ldots, x_4$
The system is consistent